Nonlinear Dynamics and Systems Theory, **20** (2) (2020) 168–178

Analysis of the Dynamics of a Two-Degree-of-Freedom Nonlinear Mechanical System under Harmonic Excitation

Akram Khalil Cheib [1,2], V. E. Puzyrov [2] and N. V. Savchenko [3*]

[1] *The International University of Beirut, Beirut, Lebanon*
[2] *Vasyl Stus Donetsk National University, Vinnytsia, Ukraine*
[3] *Zhukovsky National Aerospace University "KhAI", Kharkiv, Ukraine*

Received: November 14, 2018; Revised: March 10, 2020

Abstract: In this paper the 2-degree-of-freedom mechanical system under the action of external harmonic excitation is considered. The system consists of the rotating elastically mounted frame and attached mass (absorber) with viscous friction and nonlinear stiffness. The stability problem of periodic regimes is investigated based on the averaging method. The influence of nonlinear component is analyzed with respect to responses of the main mass in the vicinity of resonance frequencies.

Keywords: *harmonic excitation; averaging technique; resonance frequency; stability; nonlinear stiffness.*

Mathematics Subject Classification (2010): 34C46, 34D20, 70E50, 70E55, 70K20.

1 Introduction

The problems associated with unwanted vibrations are encountered in many applied tasks in machine-building, construction, aerospace engineering, biomechanics, etc. For a number of reasons, a structure may encounter excitation sources that are not provided for in the design. To improve the reliability of the design, the engineers aim at a simple, low cost and efficient solution. In many cases, dynamic vibration absorbers (DVA) meet these requirements. Dynamic vibration absorbers or tuned mass dampers are small mass-spring-damper elements locally attached to the structure designed to dissipate excessive vibration energy.

* Corresponding author: mailto:nina_savchenko@hotmail.com

One of the important goals of the vibration control is to create a frequency zone free from resonance (or a spectral gap around an inconvenient frequency) by connecting a vibration absorber. As a rule, the DVA parameters are determined according to the eigenvector of the unstable vibration mode, which ensures the spatial distribution of the vibration energy within one vibration mode. However, this single-mode approach does not take into account the influence of adjacent oscillation modes which may become important under some circumstances.

In recent decades, the nonlinear absorbers have become widespread in the passive vibration control [1 − 10]. A key feature of such DVAs is the absence of a preferred frequency, that is, they can be functional at almost any frequency. This gives a great advantage over the linear absorbers. At the same time, the nonlinearity leads to the amplitude dependency, because a critical energy threshold exists, and an "inappropriate" nonlinear characteristic of the DVA generates the instability of the working regime.

In the present paper, we study the system which allows the rotation of the main mass in addition to the lateral motion. Such systems may be found in various applications in which a relative rotational motion is presented: rotor dynamics [11], a coupled pitch-roll ship model under harmonic excitation [12, 13], vibration control by rotating masses [14], rotor dampers in bridge structures [15], the use of the Helmholtz resonators for low frequencies in acoustics [16] and many others.

2 Statement of the Problem

Let us consider the following mechanical system: the frame of mass m_1 is mounted on the weightless platform as it is schematically shown in Fig. 1.

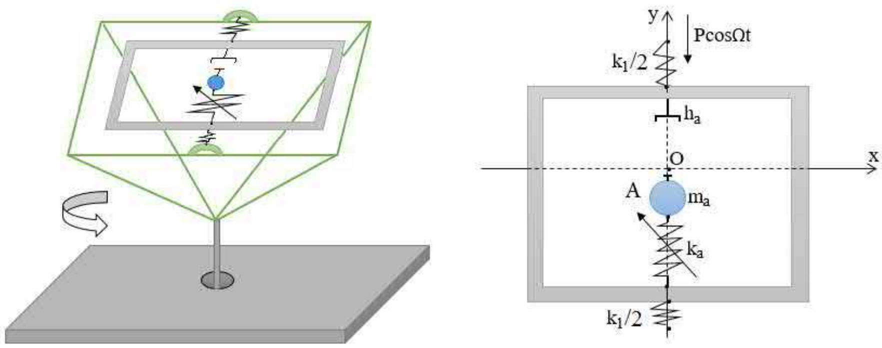

Figure 1: Elastically conjoint rotating frame with the DVA.

The platform is rotating with a constant angular velocity ω around the vertical axis. The frame is subjected to the harmonic excitation $F_e = Pcos\Omega t$. The absorber A is attached to the frame with the aim to prevent the possible resonance.

The motion equations of the mechanical system under consideration are

$$\begin{aligned} &(m_1+m_a)\ddot{y}_1 + m_a\ddot{\tilde{y}} + [k_1 - (m_1+m_a)\omega^2]y_1 - m_a\omega^2\tilde{y} = Pcos\Omega t, \\ &m_a\ddot{y}_1 + m_a\ddot{\tilde{y}} + h_a\dot{\tilde{y}} - m_a\omega^2 y_1 + (k_a - m_a\omega^2)\tilde{y} - k_3\tilde{y}^3 = 0. \end{aligned} \tag{1}$$

The coordinate y represents the displacement of the main mass m_1 with respect to its frame, while $\tilde{y}$ stands for the relative displacement of the absorber mass m_a with respect to the mass m_1. The stiffness of the main mass is denoted by k_1, and the restoring force of the absorber is expressed as $k_a\tilde{y} - k_3\tilde{y}^3$; h_a is the coefficient of viscous friction.

Introducing the dimensionless parameters and time by the formulas

$$\mu = \sqrt{\frac{m_a}{m_1}},\ \kappa_1 = \frac{k_1 - (m_1 + m_a)\omega^2}{m_1\Omega^2},\ \alpha = \frac{\omega^2}{\Omega^2},\ p = \frac{P}{m_1\Omega^2},\ \ h = \frac{h_a}{m_a\Omega}, \\ \kappa_2 = \frac{k_a - m_a\omega^2}{m_a\Omega^2},\ \kappa_3 = \frac{3m_1k_3}{4m_a^2\Omega^2},\ \tau = \Omega t, \tag{2}$$

and replacing the variable $\tilde{y}$ with μy_1, we can rewrite the motion equations in the following form:

$$My'' + Dy' + Ky = F. \tag{3}$$

Here

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},\ M = \begin{pmatrix} 1+\mu^2 & \mu \\ \mu & 1 \end{pmatrix},\ D = diag(0, h),\ K = \begin{pmatrix} \kappa_1 & -\mu\alpha \\ -\mu\alpha & \kappa_2 \end{pmatrix},$$

$$F = \begin{pmatrix} pcos\tau \\ \frac{4}{3}\kappa_3 y_2^3 \end{pmatrix}$$

and prime means the derivative with respect to time τ.

Let us introduce the complex variables

$$z = \mathrm{col}(z_1, z_2),\ z_j = (y_j + \imath y_j')e^{\imath\tau},\ j = 1, 2.$$

Taking into account that

$$y = Re(e^{-\imath\tau}\ z),\ y' = Im(e^{-\imath\tau}\ z),\ y'' = Im(e^{-\imath\tau}\ (z' - \imath z)),$$

equations (3) can be rewritten as follows:

$$MIm[e^{-\imath\tau}(z' - \imath z)] + DIm(e^{-\imath\tau}z) + KRe(e^{-\imath\tau}z) = F. \tag{4}$$

In addition to equation (4), the relation

$$Re[e^{-\imath\tau}(z' - \imath z)] - Im(e^{-\imath\tau}z) = 0 \tag{5}$$

holds (because of $y' = \frac{d}{d\tau}y$).

Multiplying both sides of equality (5) by M from the left-hand side and adding the equation (4) with the multiplier $\imath$, we get

$$e^{-\imath\tau}M(z' - \imath z) - (M - \imath D)Im(e^{-\imath\tau}z) + \imath KRe(e^{-\imath\tau}z) = \imath F.$$

Now we assume that z_1, z_2 are the slow functions in time τ. Applying the method of averaging [17], we get

$$2Mz' + (D + \imath C)z = \imath F_1,\ 2M\bar{z}' + (D - \imath C)\bar{z} = -\imath\overline{F}_1, \\ C = K - M = \begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{pmatrix},\ F_1 = \begin{pmatrix} p \\ \kappa_3 z_2^2\bar{z}_2 \end{pmatrix}. \tag{6}$$

It should be noted that

$$\langle e^{\imath\tau}Im(e^{-\imath\tau}z)\rangle = -\frac{1}{2}\imath z,\ \langle e^{\imath\tau}Re(e^{-\imath\tau}z)\rangle = \frac{1}{2}z,\ \langle e^{\imath\tau}(Re(z)^3\rangle = \frac{3}{8}z^2\bar{z},$$

where
$$\langle * \rangle \triangleq \frac{1}{2\pi}\int_0^{2\pi}(*)d\tau.$$

The conditions for periodic steady state vibration are determined by stationary points of the system (6), namely,

$$c_{11}z_1 + c_{12}z_2 = p,\ c_{12}z_1 + (c_{22} - \imath h)z_2 = \kappa_3 z_2^2 \bar{z}_2, \tag{7}$$

and their conjugate counterparts for $\bar{z}_1, \bar{z}_2$.

Depending on the value of c_{11}, we have two cases.

1) $c_{11} = 0$. This case may be interpreted as a resonant case for the system with "frozen" absorber, i.e., $\tilde{y} = 0, \dot{\tilde{y}} = 0$, and

$$\frac{k_1 - (m_1 + m_a)\omega^2}{m_1 + m_a} = \Omega^2. \tag{8}$$

Then, taking into account that $c_{12} = -\mu(1 + \omega^2/\Omega^2) \neq 0$, from equations (7) it is easy to get

$$z_{20} = \frac{p}{c_{12}},\ z_{10} = \frac{p}{c_{12}^2}(-c_{22} + \imath h + \kappa_3\frac{p^2}{c_{12}^2}). \tag{9}$$

Hence, the system (6) has a unique constant solution (9).

2) $c_{11} \neq 0$. Then expressing z_1 from the first equation (7) and substituting it into the second one, we get

$$c_{12}p + (\delta - \imath h c_{11})z_2 = c_{11}\kappa_3 z_2^2\bar{z}_2, \quad c_{12}p + (\delta + \imath h c_{11})\bar{z}_2 = c_{11}\kappa_3 z_2 \bar{z}_2^2, \tag{10}$$

where $\delta = \det C$. Subtracting the second equation (10) from the first one, we come to the relation

$$(\delta - c_{11}\kappa_3 z_{20}\bar{z}_{20})(z_{20} - \bar{z}_{20}) - \imath h c_{11}(z_{20} + \bar{z}_{20}) = 0. \tag{11}$$

Also we subtract the second equation (10) with the multiplier z_{20} from the first equation multiplied by $\bar{z}_{20}$. As a result, we have another auxiliary equation

$$pc_{12}(z_{20} - \bar{z}_{20}) + 2\imath h c_{11} z_{20}\bar{z}_{20} = 0. \tag{12}$$

Denoting the real and imaginary parts of z_{20} by u_0 and v_0, respectively, from (11) and (12) we get the following system:

$$[\delta - c_{11}\kappa_3(u_0^2 + v_0^2)]v_0 - hc_{11}u_0 = 0, \quad pc_{12}v_0 + hc_{11}(u_0^2 + v_0^2) = 0, \tag{13}$$

which is equivalent[1] to the system (7). These last equations allow the variable u_0 to be expressed in terms of v_0 : $u_0 = v_0(h\delta + pc_{12}\kappa_3 v_0)/h^2 c_{11}$, which leads to the cubic equation

$$\Phi(v_\star) = v_\star^3 + 2\delta v_\star^2 + (\delta^2 + h^2 c_{11}^2)v_\star + p^2 c_{11} c_{12}^2 \kappa_3 = 0,\ v_\star = \frac{p}{h}c_{12}\kappa_3 v_0.$$

The number of the real roots of this equation is determined by the sign of discriminant of the polynomial $\Phi(v)$. Namely, the latter has three different real roots if the expression

$$D_\Phi = -c_{11}[27c_{11}(p^2c_{12}^2\kappa_3)^2 - 4\delta(\delta^2 + 9h^2c_{11}^2)p^2c_{12}^2\kappa_3 + 4h^2c_{11}(h^2 + 2\delta^2)^2] \tag{14}$$

[1] With the proviso that $z_{20} \neq 0$. As $c_{12} \neq 0$, it is obvious that otherwise the equalities (7) are broken.

is positive, and it has one real root if D_Φ is negative. The expression (14) can be considered as a polynomial of the second order with respect to the parameter κ_3, hence, the necessary and sufficient condition for D_Φ to be positive is the following double inequality:

$$\frac{2}{27}\frac{\delta(\delta^2+9h^2c_{11}^2)-(\delta^2-3h^2c_{11}^2)^{3/2}}{p^2c_{11}c_{12}^2} < \kappa_3 < \frac{2}{27}\frac{\delta(\delta^2+9h^2c_{11}^2)+(\delta^2-3h^2c_{11}^2)^{3/2}}{p^2c_{11}c_{12}^2}. \tag{15}$$

Thus, the necessary condition for three real roots of $\Phi(v)$ is $d=\delta^2-3h^2c_{11}^2>0$. This condition is not so simple, with respect to formulas (2) it turns to

$$\begin{aligned}&m_1^2m_a^2(1+\alpha)^4\Omega^8-(1+\alpha)^2\Omega^6\{3(m_1+m_a)^2h_a^2+2m_1m_a(1+\alpha)[m_ak_1+(m_1+m_a)k_a]\}+\\&+(1+\alpha)\Omega^4\{6k_1(m_1+m_a)h_a^2+(1+\alpha)[m_a^2k_1^2+2m_ak_1k_a(2m_1+m_a)+k_a^2(m_1+m_a)^2]\}-\\&-k_1\Omega^2\{k_1h_a^2+k_a(1+\alpha)[m_ak_1+(m_1+m_a)k_a]\}+k_1^2k_a^2>0.\end{aligned}$$

In the case when the upper limit for the frequency value Ω is unknown, the expression for d may be positive for any linear characteristics of the absorber ($k_a,\ h_a$). Then, for the appropriate values of κ_3, which characterizes the nonlinear component of absorber's stiffness, the polynomial $\Phi(v)$ has three real roots, and the system (7) has three stationary points.

Alternatively, if the upper limit for the excitation frequency is bounded from above, then with the condition $h>|\delta/\sqrt{3}c_{11}|$, the system (7) has the unique stationary point. The same result takes place in the vicinity of condition (8), because of $c_{11}\to 0,\ \delta\to -c_{12}^2$, and the right-hand side of double inequality (15) tends to $-\infty$, though $D_\Phi<0$.

3 Stability Analysis

In order to determine the stability of a periodic solution related to averaged equations (6), the small perturbations of the solutions are introduced in a common way

$$z(\tau)=z_0+\tilde{z}(\tau),$$

where z_0 is a solution of the system (7). Taking into account that

$$\frac{\partial F_1}{\partial z}=\kappa_3\begin{pmatrix}0&0\\0&2z_2\bar{z}_2\end{pmatrix},\quad \frac{\partial F_1}{\partial \bar{z}}=\kappa_3\begin{pmatrix}0&0\\0&z_2^2\end{pmatrix},$$

the $\lambda-$ matrix for linearized system (6) has the following form:

$$\begin{pmatrix}2\lambda+\imath c_{11}&\imath c_{12}&0&0\\\imath c_{12}&2\lambda+h+\imath(c_{22}-\kappa_3z_{20}\bar{z}_{20})&0&-\imath\kappa_3z_{20}^2\\0&0&2\lambda-\imath c_{11}&-\imath c_{12}\\0&\imath\kappa_3\bar{z}_{20}^2&-\imath c_{12}&2\lambda+h-\imath(c_{22}-\kappa_3z_{20}\bar{z}_{20})\end{pmatrix}.$$

Accordingly, the characteristic polynomial is as follows:

$$f(\lambda)=a_4\lambda^4+a_3\lambda^3+a_2\lambda^2+a_1\lambda+a_0,$$

where

$$\begin{aligned}&a_4=16,\ a_3=16h(1+\mu^2),\ a_2=4\{c_{11}^2+2c_{12}^2+c_{22}^2-4\mu c_{12}[c_{11}+c_{22}(1+\mu^2)]+\\&+2\mu^2(c_{11}c_{22}+c_{22}^2+2c_{12}^2)+\mu^4c_{22}^2+h^2(1+\mu^2)^2\}-8a[\mu^2c_{11}-2\mu(1+\mu^2)c_{12}+\\&+(1+\mu^2)^2c_{22}]+3(1+\mu^2)^2a^2, a_1=4h[(c_{11}-\mu c_{12})^2+c_{12}^2],\\&a_0=\delta^2+h^2c_{11}^2-2c_{11}\delta\sigma+\frac{3}{4}c_{11}^2\sigma^2.\end{aligned} \tag{16}$$

Here the relation $\sigma = \kappa_3 z_{20}\bar{z}_{20} = \kappa_3(u_0^2 + v_0^2)$ is introduced. The solution under study is asymptotically stable if all roots of the polynomial $f(\lambda)$ have negative real parts. Such conditions, according to the Lienard–Chipart criterion [18], can be written in the form

$$a_0 > 0,\ a_2 > 0,\ \Delta_3 = \begin{vmatrix} a_3 & a_1 & 0 \\ a_4 & a_2 & a_0 \\ 0 & a_3 & a_1 \end{vmatrix} > 0. \tag{17}$$

Note the fact that in our case the condition $a_2 > 0$ is excessive. In fact, because of $h > 0,\ c_{12} \neq 0$, the coefficients a_1, a_3 are positive. But then, if $a_2 \leq 0$, we have $\Delta_3 = -[a_0 a_3^2 + a_4 a_1^2 + a_1 a_3(-a_2)] < 0$. Thus, the inequality $\Delta_3 > 0$ entails the positiveness of a_2.

The boundary cases $a_0 = 0,\ \Delta_3 = 0$ determine the bifurcation surfaces in parameters space. Namely:

A) $a_0 = 0,\ \Delta_3 > 0$. The polynomial $f(\lambda)$ has one zero root and three roots with negative real parts;

B) $a_0 > 0,\ \Delta_3 = 0$. There are two purely imaginary roots

$$\lambda_{1.2} = \pm\imath\frac{1}{2}\sqrt{\frac{(c_{11} - \mu c_{12})^2 + c_{12}^2}{1 + \mu^2}}$$

and two roots with negative real parts;

C) $a_0 = 0,\ \Delta_3 = 0$. The polynomial $f(\lambda)$ has one zero root, a pair of purely imaginary roots $\lambda_{1,2}$ and one negative real root.

In Fig. 2 the typical form of bifurcation surface $\Delta_3 = 0$ related to the parameters κ_2, σ, h is presented. After substituting the expressions from (16), we have

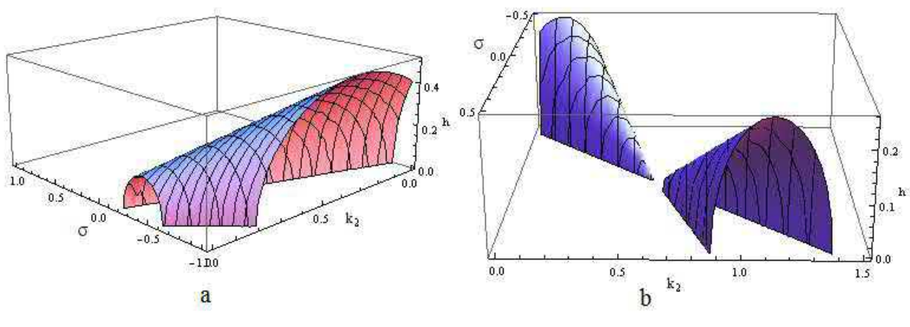

Figure 2: The typical form of bifurcation surface $\delta_3 = 0$. Cases a and b are related to the values of $\kappa_1 = 0.5$ and $\kappa_1 = 1.2$, respectively.

$$\Delta_3 = 64h^2[c_{11}\mu - c_{12}(1+\mu^2)]^2\{3\sigma^2(1+\mu^2)^2 - 8\sigma(1+\mu^2)[c_{11} - 2\mu c_{12} + c_{22}(1+\mu^2)] + \\ + 4h^2(1 + \mu^2 + \mu^4 c_{11}^2) + 4[c_{11} - 2\mu c_{12} + c_{22}(1+\mu^2)]^2\}.$$

The expression in square brackets can be rewritten as

$$c_{11}\mu - c_{12}(1+\mu^2) = \mu(\kappa_1 - 1 - \mu^2) + \mu(1+\alpha)(1+\mu^2) = \mu[\kappa_1 + \alpha(1+\mu^2)].$$

Due to the assumption $\kappa_1 > 0$, it is positive, therefore the sign of Δ_3 is determined by the sign of the expression in brackets. The latter can be presented in the form

$$\{*\} = 4h^2(1+\mu^2+\mu^4 c_{11}^2) + (3O_1 - 2Q_2)(Q_1 - 2Q_2),\ Q_1 = \sigma(1+\mu^2),$$
$$Q_2 = c_{11} - 2\mu c_{12} + c_{22}(1+\mu^2).$$

For the linear absorber we have $\kappa_3 = 0$, hence $\sigma = 0$, and $\{*\} > 0$ (as well as $a_2 > 0$, $a_0 > 0$). Then, if the parameter κ_3 is rather small[2], the inequalities (17) are still valid.

4 Numerical Simulations and Discussion

Although it is possible to determine the type of the stationary points of the system in explicit form relatively to the set of parameters $\{m_1, k_1, m_a, k_a, \omega, h, P, \Omega\}$, it is reasonable not to do this. Firstly, due to relations (12), (13) the explicit representation of the expression for σ is too cumbersome and not convenient for analysis. Another reason is that for the purposes of engineering applications, it is much more suitable to use the procedure oriented towards numerical calculations, which allow to do some significant simplifications (for instance, introduce one or several small parameters [3]).

The calculations were carried out as follows: for some given values of μ, α and κ_1, the magnitudes of κ_2 and h were determined as for the linear DVA according to [19, 20]. Then the values found (say, κ_{20}, h_0) were changed, and the responses of the main mass were analyzed. Thereafter, the influence of the nonlinear stiffness k_3 was investigated. The results are presented in Figs. 3-6. The parameter p may be counted as optional, because with the transformation

$$y = py_\star, \quad \tilde{y} = p\tilde{y}_\star, \quad k_3 = \frac{k_{3\star}}{p^2}$$

one can add the subscript "$\star$" to the variables, their derivatives and parameter k_3, and take $p = 1$. The magnitude of P/Ω^2 may be small enough, hence the initial conditions for integrating the motion equations (1) were taken correspondingly. For the reason of simplicity, the subscript "$\star$" is omitted in the figures.

In Fig. 3 the evolution of oscillations of the main mass is presented. For $\mu = 0.2$, $\alpha = 0.1, k_1 = 1.25$ and the fixed value of $k_2 = k_{20} \approx 1.17$, the magnitude of h_a was changed. Based on the obtained calculations, we can conclude the following.

1) If the coefficient h is small ($h < 0.05$ in our example), the amplitude of oscillations is increasing slowly. Although in linear case the motion is asymptotically stable in the Lyapunov sense, in the vicinity of resonant frequencies small denominators in the general solution of system (1) (with $k_3 = 0$) appear, and the damping rate cannot "counteract" successfully to the growth of the amplitude. This growth is not unbounded, but it is big enough (Fig.3*a*), so such absorber is ineffective.

2) The optimal magnitude of damping coefficient belongs to some "middle" range ($h \in [0.08, 0.12]$). The behaviour of solutions is characterized as follows. After the initial perturbations, there exists a transitional time interval (about $100 - 120$ τ- seconds, see Fig.3) where the amplitude achieves its maximal value, and varying the parameters of

[2] Or p is small, which leads to the smallness of v_0, u_0 due to formulas (12), (13).

[3] Depending on the circumstances, it may be $\mu, p, 1 - \frac{k_1}{m_1\Omega^2}, h$ and so on.

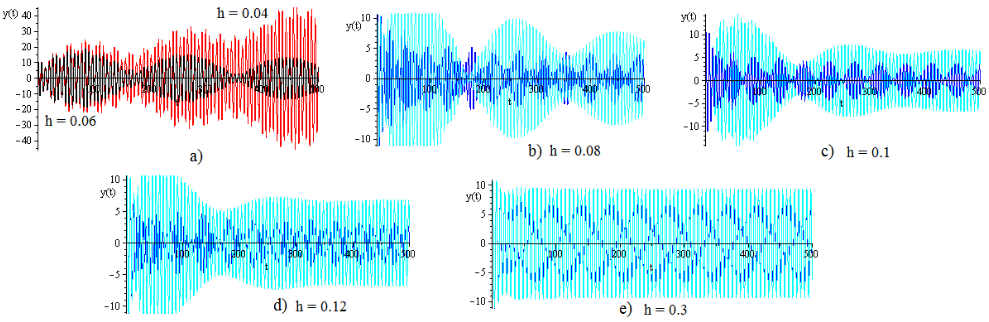

Figure 3: The shape of the oscillations of the main mass depending on the damping coefficient h. The light lines (cyan) correspond to the value $k_3 = 0$, the dark ones (blue) – to the value $k_3 = 0.0002$.

DVA has little effect on reducing this value. However, after this period, we can distinguish two cases:

A) On the left side of this interval the amplitude of oscillations goes down slowly, but finally, the smallest rate of the main mass responses is achieved (Fig. $3b, c$).

B) Increasing the value of h over 0.1 leads to more fast mitigation of the oscillation amplitude, but the rate of the main mass responses grows up (Fig. $3d$).

3) Finally, with further increase of the parameter h, the shape of the oscillations changes (Fig.3e). In contradiction to the previous cases, this shape becomes of the "stripe" type - like for a simple harmonic oscillation – for both cases $k_3 = 0$ and $k_3 = 0.0002$ (increasing the value of k_3 over 0.0008 leads to instability). The amplitude of oscillations is big enough, and such values of the damping coefficient, as well as too small values in case 1), are unsuitable for mitigation of the responses of the system.

These results were compared with the integration of averaged equations transformed by the substitution $u_j = pr_j cos\varphi_j, v_j = pr_j sin\varphi_j$ $(j = 1, 2)$. These equations are

$$2r_1' - r_2[\mu hcos(\varphi_2 - \varphi_1) + (\tilde{c}_{12} + \mu\kappa_{3\star}r_2^2)sin(\varphi_2 - \varphi_1)] = sin\varphi_1,$$

$$2r_1\varphi_1' + \tilde{c}_{11}r_1 + r_2[-\mu hsin(\varphi_2 - \varphi_1) + (\tilde{c}_{12} + \mu\kappa_{3\star}r_2^2)cos(\varphi_2 - \varphi_1)] = cos\varphi_1,$$

$$2r_2' + \tilde{c}_{21}r_1 sin(\varphi_2 - \varphi_1) + h(1 + \mu^2)r_2 = -\mu sin\varphi_2,$$

$$2r_2\varphi_2' + r_2[\tilde{c}_{22} - \kappa_{3\star}(1 + \mu^2)r_2^2] + \tilde{c}_{21}r_1 cos(\varphi_2 - \varphi_1) = -\mu cos\varphi_2.$$

Here $\tilde{c}_{jk}$ are the elements of the matrix $M^{-1}C$.

The time histories for $r_1(t)$ are presented in Fig. 4.

Also, the influence of the coefficient κ_3 (which relates to nonlinear stiffness) was tested. The results are presented in Fig. 5 for given values $\mu = 0.2, \kappa_1 = 0.95, \kappa_2 = 0.97$, $\alpha = 0.1, h = 0.25$. Both cases $\kappa_3 > 0$ (softening spring) and $\kappa_3 < 0$ (hardening spring) were considered. The hardening spring (line 1 in Fig. 5) gives the worst result, while the softening spring with $\kappa_3 \approx 0.006$ (line 4) gives the best response. The further increasing of κ_3 (line 5) worsens the state.

Finally, in Fig.6 the mutual influence of the linear and nonlinear characteristics of the restoring force on the responses of the main mass is presented.

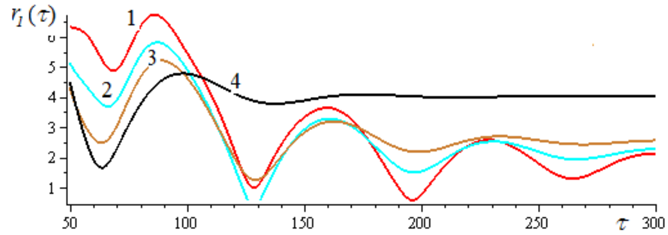

Figure 4: Time history for $r_1(\tau)$. Lines 1 - 4 are related to the values of $h = 0.08, 0.1, 0.12$ and 0.15, respectively.

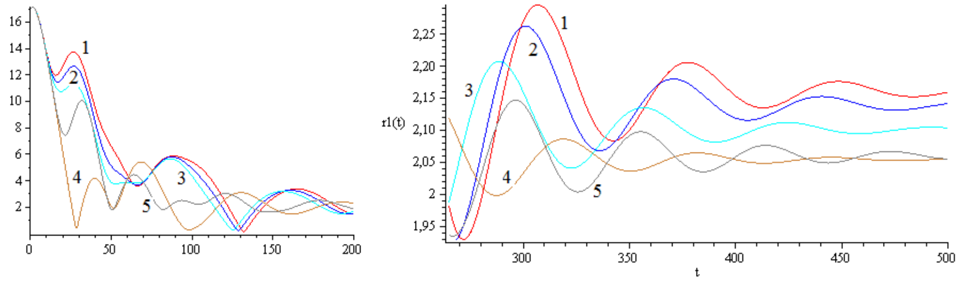

Figure 5: Influence of the nonlinear stiffness on the responses of the main mass. The values of κ_3 for lines 1 – 5, respectively, are $-0.002, 0.0, 0.002, 0.006, 0.008$.

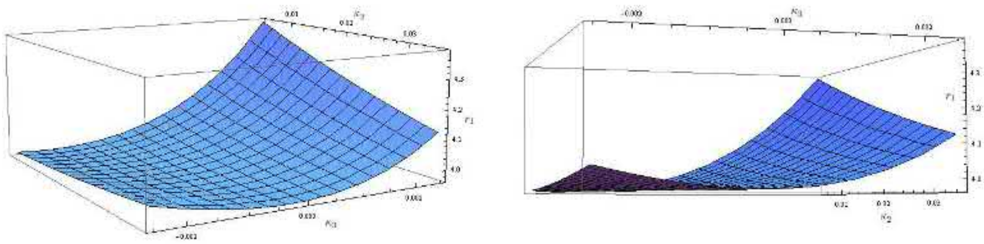

Figure 6: Influence of the absorber's stiffnesses on the responses of the main mass.

5 Conclusion

In this paper we have studied the dynamics of a 2-DOF mechanical system with combined translational and rotational motions under external harmonic excitation. The influence of the dynamical absorber with nonlinear stiffness on the responses of the main mass was investigated. The analytical approach based on the averaging method was used. The stability conditions for periodic solutions are obtained and analyzed. The numerical calculations have shown that the averaged equations correlate well with the motion equations. A suitable choice of the absorber's parameters (linear and nonlinear stiffness, and the damping coefficient) in the vicinity of resonant frequencies was discussed. In particular, it was shown that the nonlinear spring of the absorber may improve essentially the responses of the main mass (counteract to the growth of the amplitude) caused by the external excitation.

References

[1] O. Gendelman, L. Manevitch, A. Vakakis and R. M'Closkey. Energy pumping in nonlinear mechanical oscillators. *Journal of Applied Mechanics* **68** (2001) 34–41.

[2] A. Vakakis and R. Rand. Non-linear dynamics of a system of coupled oscillators with essential stiffness non-linearities. *International Journal of Non-Linear Mechanics* **39** (2004) 1079–1091.

[3] S. J. Zhu, Y. F. Zheng and Y. M. Fu. Analysis of non-linear dynamics of a two-degree-of-freedom vibration system with non-linear damping and non-linear spring. *Journal of Sound and Vibration* **271** (2004) 15–24.

[4] O. Gendelman, Y. Starosvetsky and M. Feldman. Attractors of harmonically forced linear oscillator with attached nonlinear energy sink I: Description of response regimes. *Nonlinear Dynamics* **51** (2008) 31–46.

[5] R. A. Ibrahim. Recent advances in nonlinear passive vibration isolators. *Journal of Sound and Vibration* **314** (2008) 371–452.

[6] L. Kela and P. Vhoja. Recent Studies of Adaptive Tuned Vibration Absorbers/Neutralizers. *Applied Mechanics Reviews* **62** (6) (2009) 060801–060810.

[7] J. C.Ji and N. Zhang. Suppression of the primary resonance vibrations of a forced nonlinear system using a dynamic vibration absorber. *Journal of Sound and Vibration* **329** (2010) 2044–2056.

[8] H. El-Ghareeb Taha, S. Hamed Yaser and S. Abd Elkader Mohamed. Non-Linear Analysis of Vibrations of Non-Linear System Subjected to Multi-Excitation Forces via a Non-Linear Absorber. *Applied Mathematics* **3** (2012) 64–72.

[9] M. Makkar and J.-Y. Dieulot. Passivity Based Control of Continuous Bioreactors. *Nonlinear Dynamics and Systems Theory* **17** (4) (2017) 357–368.

[10] G. I. Melnikov, N. A. Dudarenko, K. S. Malykh, L. N. Ivanova and V. G. Melnikov. Mathematical Models of Nonlinear Oscillations of Mechanical Systems with Several Degrees of Freedom. *Nonlinear Dynamics and Systems Theory* **17** (4) (2017) 369–375.

[11] M. Guskov, J.-J. Sinou and F. Thouverez. Multi-dimensional harmonic balance applied to rotor dynamics. *Mechanics Research Communications* **35** (2008) 537–545.

[12] R. Pan and H. G. Davies. Responses of a non-linearly coupled pitch-roll ship model under harmonic excitation. *Nonlinear Dynamics* **9** (4) (1996) 349–368.

[13] M. Sayed and Y. S. Hamed. Stability and response of a nonlinear coupled pitch-roll ship model under parametric and harmonic excitations. *Nonlinear Dynamics* **64** (3) (2011) 207–220.

[14] R. Baumer and U. Starossek. Active vibration control using centrifugal forces created by eccentrically rotating masses. *J. Vib. Acoust. ASME* **138** (4) (2016) 1–14.

[15] Y. Zhang, L. Li and X. Zhang. Switch Control of Twin Rotor Damper for Bridge Vibration Mitigation under Different Excitations. *X International Conference on Structural Dynamics EURODYN 2017, Procedia Engineering* **199** (2017) 1707–1712.

[16] S. Bellizzi, R. Cote and M. Pachebat. Responses of a two degree-of-freedom system coupled to a nonlinear damper under multi-forcing frequencies. *Journal of Sound and Vibration, Elsevier* **332** (7) (2013) 1639–1653.

[17] Yu. A. Mitropolskii. *Method of Averaging in the Investigations of Resonance Systems.* Moscow: Nauka, 1992. [Russian]

[18] A. Lienard and M. H. Chipart. Sur le signe de la partie reelle des racines d'une equation algebrique. *J. Math. Pures Appl.* **10** (6) (1914) 291–346.

[19] C. P. Den Hartog. *Mechanical Vibrations*. New York: McGraw-Hill, 1956.

[20] A. Thompson. Optimum tuning and damping of a dynamic vibration absorber applied to a force excited and damped primary system. *Journal of Sound and Vibration* **77** (1981) 403–415.