Stability and Boundedness of the Solutions of Multi-Parameter Dynamical Systems with Circulatory Forces

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Abstract: We consider a linear dynamical system under the action of potential and circulatory forces. The matrix of potential forces is positive definite, and the main question is when the circulatory forces induce instability to the system. Different approaches to studying the problem are discussed and illustrated by examples. The case of multiple eigenvalues also is considered, and sufficient conditions of instability are obtained. Some issues of the dynamics of a nonlinear system with an unstable linear approximation are discussed. The behavior of trajectories in the case of unstable equilibrium is investigated, and an example of the chaotic behavior versus the case of bounded solutions is presented and discussed.

1. Introduction

In the theory of dynamical systems, interest in the properties of symmetry is steadily growing. In recent decades, a systematic study of systems possessing the so-called reversible symmetry has become widespread [1–5]. Consider an autonomous dynamical system with continuous time, governed by the ordinary differential equation:

$$\frac{d\boldsymbol{x}}{dt} = f(\boldsymbol{x}). \tag{1}$$

It is said [1] that the map $\boldsymbol{S} : \mathbb{R}^n \to \mathbb{R}^n$ is a reversing symmetry of (1) if:

$$\frac{d}{dt}(\boldsymbol{S}\boldsymbol{x}) = -f(\boldsymbol{S}\boldsymbol{x}).$$

In this case, System (1) is invariant with respect to transformation $(t, \boldsymbol{x}) \to (-t, \boldsymbol{S}\boldsymbol{x})$.

The stability problem for non-conservative dynamical systems is one of the fundamental problems in modern science and technology. It is related to many real-life applications and is mainly matched with the response of mechanical systems to parametric excitation and flutter phenomenon exhibited by aerospace systems undergoing interaction between a structure and fluid flows, as well as playing a crucial role in the control of walking robots and many others [6–11].

In this paper, we study a generic canonical form of a finite-dimensional system governed by the second order ODEs presented in the Euler–Lagrange form:

$$\frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{q}_j} - \frac{\partial \mathcal{K}}{\partial q_j} = \frac{\partial \Pi}{\partial q_j} + Q_j \ (j = 1, \cdots, n),$$

where $\mathcal{K}, \Pi$ are the kinetic and potential energies of the system, respectively, and Q_j represent the generalized non-potential forces. The dynamics of such a system in the vicinity of equilibrium is described by linearized equations:

$$M\ddot{x} + (D + G)\dot{x} + (K + N)x = 0, \tag{2}$$

which is often referred to as the MDGKNsystem [12,13]. The symmetrical matrices M, D, K stand for the inertia, dissipative, and stiffness matrices, respectively, and the skew-symmetric ones G, N are associated with gyroscopic and non-conservative positional or circulatory forces.

A "strange" property of some non-conservative systems is that the influence of damping can make the system unstable. This property is often referred to as Ziegler's paradox [14], since adding a viscous damping to the joints of a double pendulum subjected to the follower force destabilizes the system. After the publication of Ziegler in 1952, investigations of the destabilizing effect of damping have attracted the attention of numerous researchers [15–20] and are continued to this day [5,21,22]. Furthermore, an additional interesting feature of the "reciprocal" damping effect follows: when adding a damping to unstable non-conservative system, this yields stabilization of an unstable system (in the Lyapunov sense), and the further increasing of damping leads back to instability [23]. A general stability problem of equilibrium of different mechanical non-conservative (both dissipative and circulatory) systems was studied in numerous papers [12,13,24–26]. Reviews of the problem for dissipative systems may be found in the works of Bolotin [27], Langthjem and Sugiyama [28], the monograph of Kirillov [11], and the recent work by Bigoni and Kirillov [5].

However, in our study, we shall focus on the undamped non-gyroscopic systems of the *MKN*-type (which are reversible systems). The eigenvalues of such a system are symmetrically located on the complex plane with respect to the imaginary axis; thus, stability is possible only if all of them are situated on this axis. The *MKN* systems were studied in works [29–34]. Since the matrix M is positive definite and symmetric, there exists [35,36] a non-singular transformation, which transforms this system to the IK_0N form, where I is an identity matrix and K_0 has a diagonalized form. For this reason, some authors [32–34] consider the IK_0N systems. Though this simplification is justified from the theoretical point of view, often the elements of matrices are functions of physical parameters, and such a reduction is not of great utility. We shall use this form, as well as the more general form *MKN* below.

The paper is organized as follows. Section 2 describes three main approaches to the study of the stability problem of the linear circulation systems. Section 3 deals with a multi-parameter system with three degrees of freedom. To find the conditions of stability, the approaches described in Section 2 are used. Their advantages and disadvantages are compared. As an example, we consider the problem of a three link pendulum to which a tracking force is applied. Section 4 examines the special case where the system has multiple roots. Such systems are found in applications, and often, the influence of an arbitrarily small circulation force leads to the instability of the motion. In addition, for systems of high dimensionality, the determination of the stability conditions by a standard procedure may be associated with serious computational difficulties. A theorem is proven, which makes it possible to simply find sufficient conditions for the flutter instability and to detect "dangerous" zones in the system parameter space. Illustrative examples for systems with four and five degrees of freedom are given. Section 5 discusses the influence of nonlinear terms of the differential equations of motion on the system dynamics in the context of the conditions of the above theorem. It is noted that for a nonlinear system, the presence of a positive Lyapunov exponent does not necessarily lead to chaotic behavior.

Although the motion is unstable in the Lyapunov sense, it can however be bounded, which can prove to be acceptable enough in engineering applications.

2. Formulation of the Problem

The non-conservative undamped linear systems are mostly presented in the following form:

$$M\ddot{x} + (K + N)x = 0 \tag{3}$$

where M and K are positive definite symmetric real square matrices, N is real skew-symmetric, and $x \in R^n$ represents a state vector. In order to underline the absence of damping, such a system is called an undamped non-conservative or circulatory system. Because 3 is a linear autonomous ODE system, its characteristic polynomial $f(\lambda)$ has order $2n$ and contains only even degrees of λ. Consequently, the system (3) is stable if all roots of the polynomial:

$$f(\Lambda) = a_n\Lambda^n + a_{n-1}\Lambda^{n-1} + \cdots + a_1\Lambda + a_0 \tag{4}$$

are real and negative, $\Lambda = \lambda^2$. The polynomial (4) is referred to as the reduced characteristic polynomial related to (3).

Three main cases are possible:

(1) The system is marginally or weakly stable (Lyapunov stability) if all solutions $x(t)$ of (3) are bounded for $t \in [0, +\infty)$. This means that every eigenvalue λ_j is on the imaginary axis and semi-simple (there are s linearly independent eigenvectors associated with the eigenvalue of multiplicity s);

(2) The system (3) is unstable by flutter (or oscillatory unstable) if at least one of the eigenvalues λ_j is complex with the nonzero real part (there exists an oscillating motion with exponentially growing amplitude);

(3) The system (3) is unstable by divergence (or statically unstable) if all of the Λ are real and at least one of them is real positive (there is an aperiodic, exponentially growing motion).

The negativeness of the roots $\Lambda_j (j = 1, \cdots, n)$ may be validated with the help of Routh–Hurwitz or Lienard–Chipard criteria, and the absence of complex roots may be detected by at least three approaches. Below, we give their brief description.

The first approach will be referred to as the DM-approach and is based on analysis of the so called "discriminant matrix" [30,37].

Definition 1. *The discriminant matrix (in [37], the term "discrimination matrix" is used) of the polynomial (4) is the square $2n \times 2n$ matrix of the following form:*

$$D = \begin{pmatrix}
a_n & a_{n-1} & a_{n-2} & \cdots & a_0 & 0 & \cdots & 0 \\
0 & na_n & (n-1)a_{n-1} & \cdots & a_1 & 0 & \cdots & 0 \\
0 & a_n & a_{n-1} & a_{n-2} & \cdots & a_0 & \cdots & 0 \\
0 & 0 & na_n & \cdots & 2a_2 & a_1 & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdots & 0 & a_n & a_{n-1} & a_{n-2} & \cdots & a_0 \\
0 & \cdots & 0 & 0 & na_n & (n-1)a_{n-1} & \cdots & a_1
\end{pmatrix}. \tag{5}$$

Definition 2. *The sequence $\{D_1, D_2, \cdots, D_s\}$, where $D_j (j = 1, \cdots, s)$ is the leading principal minor (upper left diagonal minor) of order $2j$, is called a discriminant sequence of the polynomial (4) (minors D_j sometimes are called "sub-discriminants" or "principal sub-resultants of $f(\lambda)$ and $f(\lambda)$").*

Theorem 1. *[29,37]. Given a polynomial $f(\Lambda)$ with real coefficients, a necessary and sufficient condition for the polynomial to have only real roots is that the elements of the discriminant sequence are all non-negative $D_j \geq 0$, $j = 1, \cdots, n$.*

The second approach (B-approach [31,32]) is connected with consideration of the quadratic form $x^\tau P x$, where:

$$P = \begin{pmatrix} n & p_1 & p_2 & \cdots & p_{n-1} \\ p_1 & p_2 & p_3 & \cdots & p_n \\ p_2 & p_3 & p_4 & \cdots & p_{n+1} \\ . & . & . & . & . \\ p_{n-1} & p_n & p_{n+1} & \cdots & p_{2n-2} \end{pmatrix}, \quad p_m = (-1)^m \mathrm{Tr}(K+N)^m, \quad m = 1, \cdots, 2n-2.$$

Note that the elements p_j are connected with the coefficients of $f(\Lambda)$ by the following formulas:

$$p_j = -(p_{j-1}a_{n-1} + \cdots + p_1 a_{n-j+1} + j a_{n-j}), \; j = 1, \cdots, n. \tag{6}$$

Theorem 2. *(Bulatovic [33]). (a) The system (3) is spectrally stable (zero values for Λ are allowed) iff $P \geq 0$, and $a_j \geq 0, j = 1, \cdots, n$.*
(b) The system (3) is unstable by divergence iff $P \geq 0$, and $\exists a_j < 0$.
(c) The system (3) is unstable by flutter iff $x^\tau P x$, can take negative values.

The third approach (G-approach [35]) is based on the use of a catalecticant matrix (Hankel matrix) of a special kind. Suppose that numbers $\alpha_1, \alpha_2, \cdots, \alpha_q$ are distinct roots of the polynomial (4) with multiplicities $n_1, n_2, \cdots, n_q$ $(n_1 + n_2 + \cdots + n_q = n)$:

$$f(\Lambda) = a_n(\Lambda - \alpha_1)^{n_1}(\Lambda - \alpha_2)^{n_2} \; (\Lambda - \alpha_q)^{n_q}.$$

Introducing the Newton sums:

$$s_k = \sum_{j=1}^{q} n_j \alpha_j^k (k = 0, 1, 2, \cdots),$$

let us compose the matrix into the following form:

$$S = \begin{pmatrix} s_0 & s_1 & s_2 & \cdots & s_{n-2} & s_{n-1} \\ s_1 & s_2 & s_3 & \cdots & s_{n-1} & s_n \\ s_2 & s_3 & s_4 & \cdots & s_n & s_{n+1} \\ . & . & . & . & . & . \\ s_{n-1} & s_{n-2} & s_{n-3} & \cdots & s_{2n-3} & s_{2n-2} \end{pmatrix}_{n \times n}. \tag{7}$$

The elements of matrix S are connected with the coefficients of f by Newton recurrence relations:

$$s_0 = n, s_1 = -\frac{a_{n-1}}{a_n}, \; s_k = -\frac{1}{a_n}(a_{n-1}s_{k-1} + a_{n-2}s_{k-2} + \cdots + a_{n-k+1}s_1 +$$

$$+ a_{n-k}k) \; (k \leq n), \; s_k = -\frac{1}{a_n}(a_{n-1}s_{k-1} + a_{n-2}s_{k-2} + \cdots + a_0 s_{k-n}) \; (k > n). \tag{8}$$

For example, for $n = 3$, these formulas take the form (with $a_3 = 1$):

$$s_0 = 3, \; s_1 = -a_2, \; s_2 = -(a_2 s_1 + 2a_1) = a_2^2 - 2a_1, \; s_3 = -(a_2 s_2 + a_1 s_1 + 3a_0) =$$

$$= -a_2^3 + 3a_1 a_2 - 3a_0, \; s_4 = a_2^4 - 4a_1 a_2^2 + 4a_2 a_0 + 2a_1^2. \tag{9}$$

Theorem 3. *(Gantmakher [35]). (A) The number of roots of polynomial (4) different from each other is equal to the rank of the matrix S. (B) The number of distinct real roots of this polynomial is equal to the signature of the real quadratic form with matrix S.*

For application purposes, the following corollary is helpful:

Corollary 1. *The number of real roots of the polynomial* (4) *(with real coefficients) different from each other is equal to* $\mu_S - \nu_S$, *where* μ_S *is the "sign permanence" index and* ν_S *is the "sign variability" index (i.e., quantities of the values with the same sign and changes of the sign, respectively) in the sequence (in fact, the first element of the sequence is one, and it is followed by sequentially recorded principal minors* $\Delta_S^{(j)}$ *of matrix S till order* $2r - 2$*):*

$$
1,\ s_0,\ \begin{pmatrix} s_0 & s_1 \\ s_1 & s_2 \end{pmatrix},\ \begin{pmatrix} s_0 & s_1 & s_2 & \cdots & s_{r-1} \\ s_1 & s_2 & s_3 & \cdots & s_r \\ s_2 & s_3 & s_4 & \cdots & s_{r+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{r-1} & s_{r-2} & s_{r-3} & \cdots & s_{2r-2} \end{pmatrix}. \tag{10}
$$

Here, r is the rank of the Hankel form:

$$
S_n(x, x) = \sum_{j=1,k}^{n} s_{jk} x_j x_k, \ (s_{jk} = s_{kj} = s_{j+k-2})
$$

for polynomial $f(\lambda)$*.*

Corollary 2. *Polynomial* $f(\lambda)$ *has n distinct real roots if and only if* $r = n$, *and* $\Delta_S^j > 0, j = 2, \cdots, n$.

3. Main Results: Circulatory System with Three Degrees of Freedom

Let us consider the system (3) with the following matrices:

$$
M = I_3,\ \widetilde{K} = diag(k_1, k_2, k_3),\ \widetilde{N} = \begin{pmatrix} 0 & n_{12} & n_{13} \\ -n_{12} & 0 & n_{23} \\ -n_{13} & -n_{23} & 0 \end{pmatrix}.
$$

Firstly, we try the B-approach. The coefficients a_j are:

$$
a_3 = 1,\ a_2 = k_1 + k_2 + k_3),\ a_1 = k_1 k_2 + k_1 k_3 + k_2 k_3 + n_{12}^2 + n_{13}^2 + n_{23}^2,
$$

$$
a_0 = k_1 k_2 k_3 + k_1 n_{23}^2 + k_2 n_{13}^2 + k_3 n_{12}^2. \tag{11}
$$

The expressions for elements p_j, according to their definition, are:

$$
p_2 = \|K\|^2 - \|N\|^2,\ \ p_3 = \mathrm{Tr} K^3 - 3\mathrm{Tr} KN^2,
$$

$$
p_4 = \|K^2\|^2 + \|N^2\|^2 - 4\|KN\|^2 + 2\mathrm{Tr}((KN)^2), \tag{12}
$$

where $\| * \|$ is the Frobenius norm of a matrix $*$, i.e., $\|K\| = \sqrt{\sum_{j,s=1}^{n} k_{js}^2}$.

These formulas give some idea of how the matrices K and N influence the elements p_j, and allow describing stability/instability conditions in terms related to the structure of these matrices (for a 2-DOF system, such conditions were given in [19,31]). However, when $n \geq 3$, for computational purposes, it is more convenient to express p_4 by the coefficients of $f(\Lambda)$ and after that by p_1, p_2, p_3 according to Formula (6). Because all coefficients of $f(\Lambda)$ are positive, Case (b) of Theorem 2 is unachievable, and the semi-positiveness of matrix P defines which, Case (a) or (c), takes place. With this observation, we have two expressions to analyze: $\Delta_2 = 3p_2 - p_1^2$ and:

$$
\Delta_3 = \det P == -3p_3^2 + p_3(6p_1 p_2 - \frac{4}{3}p_1^3) - \frac{1}{6}p_1^6 - \frac{7}{2}p_1^2 p_2^2 + \frac{3}{2}p_1^4 p_2 + \frac{1}{2}p_2^3. \tag{13}
$$

Obviously, if $p_2 \leq 0$, then System (3) is unstable. Noting, in addition, that Δ_3 may be presented in the form:

$$\Delta_3 = -\frac{1}{3}(3p_3 - p_1\Delta_2 - \frac{1}{9}p_1^3)^2 + \frac{1}{54}(\Delta_2)^3,$$

and if $\Delta_2 \leq 0$, then $\Delta_3 \leq 0$ as well. This means that the positiveness of Δ_3 implies the positiveness of Δ_2, and Case (a) takes place if $p_2 > 0$, $\Delta_3 \geq 0$.

Considering the expression (13) as a polynomial of second order with respect to p_3; its positiveness leads to the following inequalities:

$$p_1 p_2 - \frac{2}{9}p_1^3 - \frac{\sqrt{2}}{18}(3p_2 - p_1^2)^{3/2} < p_3 < p_1 p_2 - \frac{2}{9}p_1^3 + \frac{\sqrt{2}}{18}(3p_2 - p_1^2)^{3/2},$$

and requirement $p_2 > p_1^2/3$ or:

$$\|K\|^2 - \|N\|^2 - \frac{1}{3}(\mathrm{Tr}K)^2 > 0$$

stands for the necessary condition of stability.

To compare the effectiveness of the B-approach, let us apply the technique of the G-approach. Taking into account the formulas (9), we get:

$$\Delta_S^{(2)} = 2(a_2^2 - 3a_1), \quad \Delta_S^{(3)} = -27a_0^2 + a_0(18a_1a_2 - 4a_2^3) - 4a_1^3 + a_1^2a_2^2.$$

Note that $\Delta_S^{(3)}$ is a quadratic polynomial on a_0 with a negative coefficient. Then, independently of other coefficients, the necessary condition of the positiveness of $\Delta_S^{(3)}$ is the non-negativeness of its discriminant (with respect to a_0): $d_1 = 16(a_2^2 - 3a_1)^3 \geq 0$. Because of this fact and taking into account that $a_1 > a_2^2/3$, both $\Delta_S^{(2)}$ and $\Delta_S^{(3)}$ are negative. This means that there is no possibility of sign sequence $\{+ + - +\}$. Therefore, if $r = 3$, consequently d_f (discriminant of $f(\lambda)$) is not equal to zero, and there are only two possible sign combinations: $\{+ + + +\}$ and $\{+ + - -\}$. In the first case, $\mu = 3, \nu = 0$, so f has three distinct real roots. The second case follows as $\mu = 2, \nu = 1$, and f has one real root and a pair of complex conjugate roots.

Eventually, we come to the single condition of stability $\Delta_S^{(3)} > 0$. Substituting the coefficients (11), we have:

$$D(f) = -4(n_{12}^2 + n_{13}^2 + n_{23}^2)^3 + n_{12}^4[3(k_1 - k_2)^2 - 2(k_1 + k_2 - 2k_3)^2] +$$

$$+ n_{13}^4[3(k_1 - k_3)^2 - 2(k_1 + k_3 - 2k_2)^2] + n_{23}^4[3(k_2 - k_3)^2 - 2(k_2 + k_3 - 2k_1)^2] +$$

$$+ \frac{1}{2}n_{12}^2 n_{13}^2[(2k_1 - k_2 - k_3)^2 + 39(k_2 - k_3)^2] + \frac{1}{2}n_{12}^2 n_{23}^2[(2k_2 - k_1 - k_3)^2 + 39(k_1 - k_3)^2] +$$

$$+ \frac{1}{2}n_{13}^2 n_{23}^2[(2k_3 - k_2 - k_1)^2 + 39(k_2 - k_1)^2] + n_{12}^2(k_1 - k_3)(k_2 - k_3)[3(k_1 - k_2)^2 -$$

$$- (2k_3 - k_1 - k_2)^2] + n_{13}^2(k_1 - k_2)(k_3 - k_2)[3(k_1 - k_3)^2 - (2k_2 - k_1 - k_3)^2] +$$

$$+ n_{23}^2(k_2 - k_1)(k_3 - k_1)[3(k_2 - k_3)^2) - (2k_1 - k_2 - k_3)^2] + (k_1 - k_2)^2(k_1 - k_3)^2(k_2 - k_3)^2.$$

The current expression is large enough for direct analysis, though it has a symmetric form (is invariant toward cyclic permutation of the indexes $1, 2, 3$). Suppose that:

$$k_1 < k_2 < k_3. \tag{14}$$

Introducing the parameter transformation:

$$n_{12}^2 = \frac{1}{4}\|N\|^4 \frac{p_1}{k_3 - k_1}, \quad n_{13}^2 = \frac{1}{4}\|N\|^4 \frac{q_2 - q_1}{k_2 - k_1}, \quad k_j = k_1 + \frac{\sqrt{2}}{4}\|N\| [b_1 + (-1)^{j+1}\sqrt{b_2}],$$

$$\lambda = \frac{\sqrt{2}}{4} \|N\| \widetilde{\lambda}, \ b_{j-1} \geq 0, \ j = 2,3; \ b_2 \leq b_1^2, \tag{15}$$

and regarding:

$$n_{23}^2 = \frac{1}{2} \|N\| - n_{12}^2 - n_{13}^2,$$

we come to the following expression for discriminant (with respect to $\widetilde{\lambda}$):

$$D = -27 p_2^2 + \frac{b_1}{2} p_2 (b_1^2 - 9 b_2 + 36) - \frac{1}{16}(4 - b_2)(b_1^2 - b_2 + 4)^2. \tag{16}$$

Being considered as the quadratic polynomial with respect to p_2, discriminant D may be positive only if expression:

$$D_{p_2} = b_1^2 (b_1^2 - 9 b_2 + 36)^2 - 27(4 - b_2)(b_1^2 - b_2 + 4)^2 = (b_1^2 + 3 b_2 - 12)^3$$

is positive. Then, the condition $D > 0$ is equivalent to:

$$\frac{1}{108}[b_1(b_1^2 - 9 b_2 + 36) - (b_1^2 + 3 b_2 - 12)^{3/2}] < q_2 < \frac{1}{108}[b_1(b_1^2 - 9 b_2 + 36) -$$
$$- (b_1^2 + 3 b_2 - 12)^{3/2}], \tag{17}$$

where parameters b_1, b_2, q_2 are connected to the coefficients of System (3) by Formula (15). The bounding surfaces from (17) are presented in Figure 1.
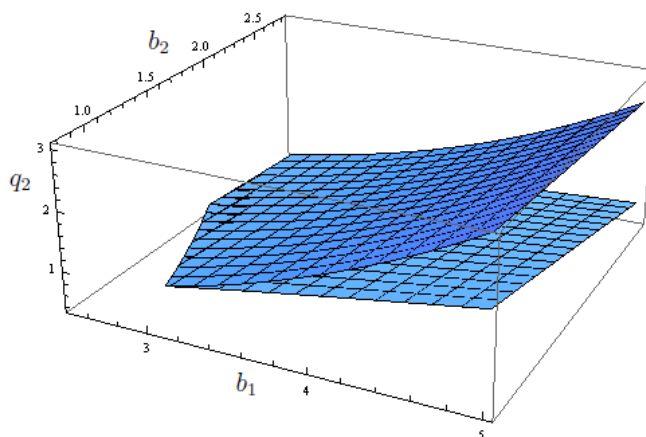

Figure 1. Domain of stability in parameters b_1, b_2, q_2.

Finally, when $d = 0$, the polynomial f has three real roots, i.e., one of multiplicity two, if $\Delta_S^{(2)} \neq 0$ ($\mu = 2$, $\nu = 0$), and of multiplicity three, if $\Delta_S^{(2)} = 0$.

To demonstrate the approach suggested, let us consider an example of a mechanical system. There are not many systems in the form of (3) with uncertain parameters for $n > 2$ in the literature (with the analytical study of stability conditions) [30,38], so in order to underline the key-points, we take an example from [30].

Example 1. *Let us consider a three link pendulum subjected to the follower force. The last has two components: tangential P_F and constant non-tangential P_C (Figure 2, non-dissipative case). In the small vicinity of equilibrium $\theta_j = 0$, $j = 1,2,3$, the linearized motion equations lead to System (3) with matrices:*

$$M = ml^2 \begin{pmatrix} 9 & 7 & 3 \\ 4 & 4 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \quad K + N = \begin{pmatrix} k - Pl & -Pl & Pl(3\alpha - 1) \\ -k & k - Pl & Pl(2\alpha - 1) \\ 0 & -k & Pl(\alpha - 1) + k \end{pmatrix}. \tag{18}$$
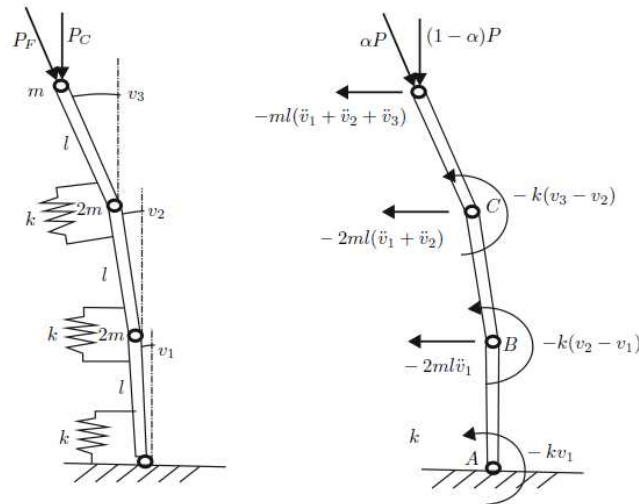
Figure 2. Dynamic model of the system subjected to a follower load (adopted from [30]).

The dimensionless characteristic equation contains two parameters: α and $\beta = Pl/k$. By the help of the MD-approach, the stability domain was obtained, and six expressions $a_2(\alpha, \beta), a_1(\alpha, \beta),$ $a_0(\alpha, \beta), D_1(\alpha, \beta), D_2(\alpha, \beta), D_3(\alpha, \beta)$ were involved in the process. Using the G-approach, only three expressions $a_2, a_2a_1 - a_3a_0,$ and $D = D_3$ are needed. In order to bring some contrast to the results of [30], we assume that $P_C = 0$ (hence, $\alpha = 1$), but at the same time, the stiffnesses of the joints k_1, k_2, k_3 differ from one another. Introducing the dimensionless parameters and time:

$$\beta = \frac{Pl}{k_1}, \ \gamma_1 = \frac{k_2}{k_1}, \ \gamma_2 = \frac{k_3}{k_1}, \ \tau = \frac{1}{l}\sqrt{\frac{k_1}{m}}\, t, \tag{19}$$

we have the following reduced characteristic polynomial:

$$f(\Lambda) = 4\lambda^3 + (2\gamma_1 + 10\gamma_2 + 14 - 8\beta)\lambda^2 + [\gamma_1\gamma_2 + 6\gamma_1 + 19\gamma_2 - 2(\gamma_1 + 3\gamma_2 + 5)\beta +$$

$$+ 3\beta^2]\lambda + \gamma_1\gamma_2.$$

The domain of stability is formed as the intersection of three sets: $a_2 > 0,$

$$g(\beta, \gamma_1, \gamma_2) = -24\beta^3 + (122 + 22\gamma_1 + 78\gamma_2)\beta^2 - 4(\gamma_1^2 + 10\gamma_1\gamma_2 + 15\gamma_2^2 + 24\gamma_1 + 84\gamma_2 +$$

$$+ 35)\beta + 2\gamma_1\gamma_2(\gamma_1 + 5\gamma_2) + 12\gamma_1^2 + 108\gamma_1\gamma_2 + 190\gamma_2^2 + 84\gamma_1 + 266\gamma_2 > 0,$$

and $D_f > 0$ (the corresponding expression is given in Appendix A).

The discriminant is the polynomial of sixth order and does not allow for any analytical reduction. Nevertheless, the corresponding 3D surfaces are smooth enough, and plotting does not present difficulties. The results are presented in Figure 3.

With the standard assumption of the equal stiffnesses at the joints $k_1 = k_2 = k_3$, the value of $\beta_{crit} \approx 1.305$ (Figure 3). With different values of k_j, this value can be markedly increased (up to 1.5 times). The optimal configuration is when k_3 is bigger and k_2 is smaller than k_1 (see Figure 4).
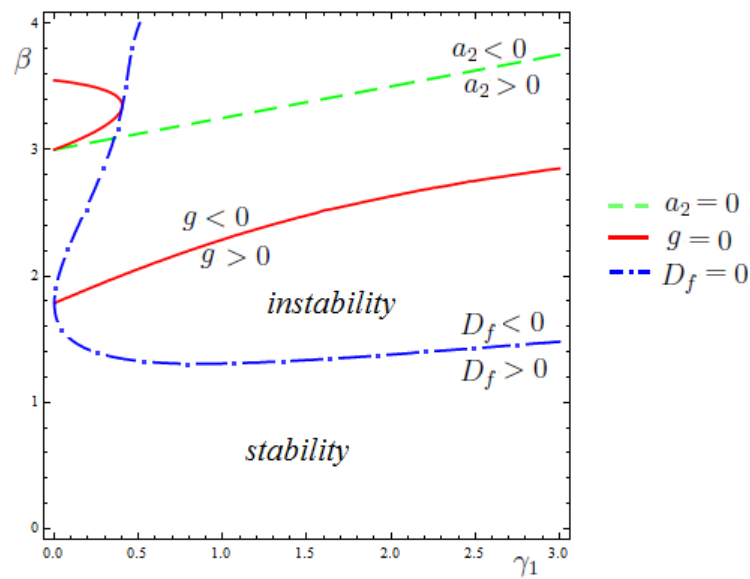
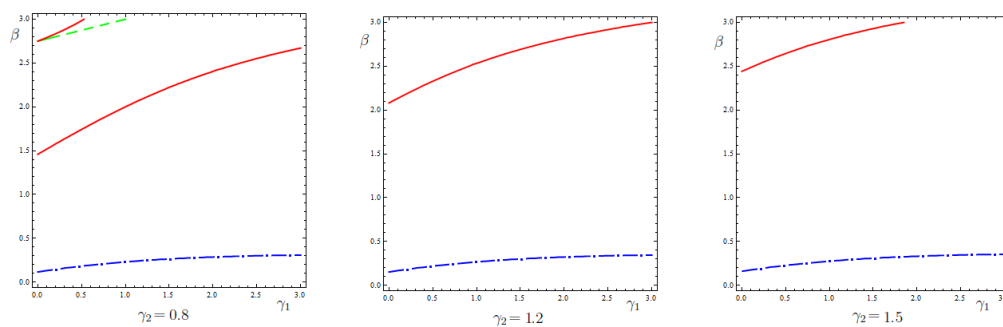**Figure 3.** Domain of stability for $\gamma_2 = 1$.

Figure 4. Variation of the stiffness values and their effect on the magnitude of the critical load.

4. Special Case: Multiple Eigenvalues of Matrix K

This section was inspired by the recent publication of F. Udwadia [34], where he generalized the well-known theorem of Merkin [24]. The last one states that when matrix K has eigenvalue Λ of multiplicity n, an arbitrary small circulatory force brings flutter instability to the system. In order to generalize this theorem, as it was pointed out by Udwadia, the commutativity of matrices K and N is required. Note that the condition of instability in case $KN = NK$ was first obtained by Bulatovic [29].

Definition 3. *[34]. Consider an n-by-n block diagonal matrix:*

$$A = \mathrm{diag}(A_1, A_2, \cdots, A_k)$$

in which the s-th (square) diagonal block A_s has dimensions i_s by i_s with $i_1 \geq i_2 \geq \cdots \geq i_k$, and another n-by-n block diagonal matrix:

$$B = \mathrm{diag}(B_1, B_2, \cdots, B_r)$$

for which the p-th (square) diagonal block B_p has dimensions j_p by j_p with $j_1 \geq j_2 \geq \cdots \geq j_r$. We shall say that matrices A and B have the same block diagonal structure if:

$$k = r, \ i_s = j_s s = 1, \cdots, k. \tag{20}$$

Theorem 4. *[34]. Let $K = \lambda_1 I_1 + \lambda_2 I_2 + \cdots + \lambda_k I_k$, and T be the orthogonal transformation such that the diagonal matrix $\Omega = T^\tau K T$ has repeated eigenvalues. The matrix $N = diag(N_1, \cdots, N_k)$ that has the same*

block diagonal structure as Ω and where $N_j, j = 1, \cdots, k$ are arbitrary skew-symmetric matrices will always commute with Ω, and the matrix $N_T = TNT^\tau$ will always commute with K. Arbitrarily small elements in the matrices N_j will lead to a flutter instability of the system $\ddot{y} + (K + N_T)y = 0$.

Though this result is interesting from a mathematical viewpoint, at first blush, it seems not to be very helpful in direct applications. The reason is that for the system (3) (as $M = I$), the requirement of the same block diagonal structure implies the decomposition of the whole system into unstable and stable (probably) subsystems. This observation yields the question: Why not consider each subsystem separately?

However, on second thought, the sense of this result opens the possibility to detect the instability for non-linear dynamical systems, especially systems of high dimension and/or multi-parameter ones. Here, the instability of a small (say, $k = 2, 3$) subsystem as a rule leads to the instability of the whole system, which may lead to serious reduction of the computational effort. Bearing this idea in mind, we shall try to introduce some modifications of the above-mentioned result based on the following way.

Let $B = (b_1, b_2, \cdots, b_n)^\tau$ be a rectangular matrix and b_j $(j = 1, \cdots, n)$ the rows. We "bisect" matrix B into two sub-matrices with q and $n - q$ rows, respectively, and shall use the notation B_q, $B \setminus B_q$ for them below.

Suppose that system MKN allows the representation:

$$M = M_1 \bigoplus M_2, \ K = K_1 \bigoplus K_2, \ N = N_1 \bigoplus N_2,$$

where matrices M_j, K_j $(j = 1, 2)$ are square symmetric and positive definite and N, N_j for skew-symmetric matrices. The dimensions of matrices M_1, K_1, N_1 are equal to $l \leq n$, and symbol "$\bigoplus$" denotes the direct sum of matrices. Let further J_{M_1} be the Jordan normal form of the matrix M_1, i.e., $M_1 = T_{M_1} J_{M_1} (T_{M_1})^{-1}$, where the columns of transformation matrix T_{M_1} are eigenvectors of M_1. If T_{M_1} is chosen in its counterpart orthogonal form, then $(T_{M_1})^{-1} = (T_{M_1})^\tau$, and the matrix:

$$\widetilde{K}_1 = (J_{M_1})^{-1/2} T_{M_1}^\tau K_1 T_{M_1} (J_{M_1})^{-1/2}$$

is symmetric and positive definite. Consequently, there exists an orthogonal matrix $T_{\widetilde{K}_1}$ such that $\widetilde{K}_1 = T_{\widetilde{K}_1} J_{\widetilde{K}_1} (T_{\widetilde{K}_1})^\tau$. The following sufficient condition of instability for system MKN is given by the following theorem.

Theorem 5. *Let the equation $det(M_1 \Lambda + K_1) = 0$ have root Λ_1 of multiplicity $q \leq l$ (in other words, matrix $M_1^{-1/2} K_1 M_1^{-1/2}$ has the eigenvalue (semi-simple) of multiplicity q). Denote by T_{M_1} and $T_{\widetilde{K}_1}$ orthogonal matrices, the columns of which are formed by the eigenvectors of the corresponding matrix, and the first q columns are associated with multiple eigenvalues. Let the matrix equality:*

$$T_q N (T \setminus T_q)^\tau = 0, \ T = T_{M_1}^\tau (J_{M_1})^{-1/2} T_{\widetilde{K}_1} \tag{21}$$

hold. Then, if:

$$N_1^\star \triangleq T_q N T_q^\tau \neq 0, \tag{22}$$

the system MKN is unstable (flutter instability).

Proof. We remind that as block matrices M_1, K_1 are symmetric and positive definite, there eigenvalues determine the transformation matrices $T_{M_1}, T_{\widetilde{K}_1}$, and:

$$M_1 = T_{M_1} J_{M_1} (T_{M_1})^{-1}, \ K_1 = T_{\widetilde{K}_1} J_{K_1} (T_{\widetilde{K}_1})^{-1},$$

where $J_{M_1} J_{K_1}$ are Jordan normal forms of those matrices.

These forms are diagonal, and without loss of generality, we may count left upper cells of dimension q as containing proportional elements corresponding to eigenvalue Λ_1. As matrices $T_{M_1}, T_{\widetilde{K}_1}$ are orthogonal, then their inverse matrices are equal to transpose ones; therefore, systems: $M_1\ddot{y} + (K_1 + N_1)y = 0$ and:

$$J_{M_1}\dot{\xi} + T_{M_1}^\tau (K_1 + N_1)T_{M_1}\xi = 0, \ \xi \in R^l \tag{23}$$

are equivalent, i.e., have the identical fundamental matrices. According to the assumption, we have $J_{\widetilde{K}_1} = \Lambda_1 I_q \oplus \mathrm{diag}(\Lambda_{q+1}, \cdots, \Lambda_l)$. Moreover, the linear nonsingular transformation $\xi = (J_{M_1})^{-1/2}T_{\widetilde{K}_1}e^{-\Lambda_1 t}\eta$ leads to the following system:

$$\dot{\eta} + (J_{K_1} - \Lambda_1 I_l + \widetilde{N}_1)\eta = 0, \tag{24}$$

and due to condition (21) $(N_1)_q = N_1^\star \oplus 0_q$, the equality holds. This means that subsystem:

$$\dot{\eta}_q + N_1^\star \eta_q = 0 \tag{25}$$

is separated. Using the Lyapunov function $V = \eta_q^\tau \eta_q$, one may see that:

$$\frac{dV}{dt}\Big|_{(25)} = -\eta_q^\tau (N_1^\star)^\tau \eta_q - \eta_q^\tau N_1^\star \eta_q = 0,$$

but because of $\mathrm{rank} N_1^\star \neq 0$, the system (25) has at least one pair of purely imaginary eigenvalues. This implies the existence of complex eigenvalues for System (24) and resulting in the flutter instability of System (3). □

Now, the calculation procedure will be explained by the following case study.

Example 2. *Consider the system (3) with:*

$$M_1 = \frac{1}{2}\begin{pmatrix} 15 & -9 & 0 & 0 \\ -9 & 15 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}, \ K_1 = 2\begin{pmatrix} 9 & -3 & -\sqrt{2} & \sqrt{2} \\ -3 & 9 & -\sqrt{2} & \sqrt{2} \\ -\sqrt{2} & -\sqrt{2} & 7 & 0 \\ \sqrt{2} & \sqrt{2} & 0 & 5 \end{pmatrix},$$

$$N_1 = \begin{pmatrix} 0 & 2\sqrt{2}n_1 & n_1 - 2n_2 & -2n_1 - n_2 \\ -2\sqrt{2}n_1 & 0 & -n_1 - 2n_2 & 2n_1 - n_2 \\ -n_1 + 2n_2 & n_1 + 2n_2 & 0 & 2\sqrt{2}n_2 \\ 2n_1 + n_2 & -2n_1 + n_2 & -2\sqrt{2}n_2 & 0 \end{pmatrix}, \tag{26}$$

where n_1, n_2 are unknown parameters.

The eigenvalues of M_1, K_1 are $12, 3, 3, 3$ and $6, 12, 18, 24$, respectively. Since the matrices $T_{M_1}, J_{M_1}, \widetilde{K}_1$ are:

$$T_{M_1} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \oplus I_2, \ (J_{M_1})^{-1/2} = \mathrm{diag}(\frac{1}{2}, 1, 1, 1), \ \widetilde{K}_1 = 2\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 6 & -2 & 2 \\ 0 & -2 & 7 & 0 \\ 0 & 2 & 0 & 5 \end{pmatrix},$$

then:

$$T_{\widetilde{K}_1} = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & 1 & 2 & -2 \\ 0 & -2 & 2 & 1 \end{pmatrix}, \ T^\tau = \frac{\sqrt{2}}{6}\begin{pmatrix} \frac{3}{2} & 2 & 1 & 2 \\ -\frac{3}{2} & 2 & 1 & 2 \\ 0 & \sqrt{2} & 2\sqrt{2} & -2\sqrt{2} \\ 0 & -2\sqrt{2} & 2\sqrt{2} & \sqrt{2} \end{pmatrix}.$$

Finally, because:

$$
\begin{pmatrix} \frac{3}{2} & -\frac{3}{2} & 0 & 0 \\ 2 & 2 & \sqrt{2} & -2\sqrt{2} \end{pmatrix} N_1 \begin{pmatrix} 1 & 1 & 2\sqrt{2} & 2\sqrt{2} \\ 2 & 2 & -2\sqrt{2} & \sqrt{2} \end{pmatrix}^\tau = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
$$

the equality (21) is satisfied, and to verify the condition (22), we get:

$$
\begin{pmatrix} 3 & -3 & 0 & 0 \\ 2 & 2 & \sqrt{2} & -2\sqrt{2} \end{pmatrix} N_1 \begin{pmatrix} 3 & -3 & 0 & 0 \\ 2 & 2 & \sqrt{2} & -2\sqrt{2} \end{pmatrix}^\tau = 54\sqrt{2}n_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
$$

According to the statement of Theorem 5, if $n_1 \neq 0$, the system (26) is unstable, independently of the value of n_2.

Remark 1. *As follows from the formulation of the theorem, it gives the sufficient condition of instability. If the condition (22) is not satisfied, the stability of the system depends on the other components of matrix N. For instance, in Example 2, the roots of $f(\Lambda)$ are:*

$$
\Lambda_{1,2} = -1 \pm \frac{\sqrt{2}}{4}in_1, \quad \Lambda_{3,4} = \frac{1}{2}(-5 \pm \sqrt{1 - 2n_2^2}),
$$

and if $n_1 = 0$, the system (26) is stable when $n_2^2 < 1/2$ (is spectrally stable when $n_2 = \pm\sqrt{2}/2$).

Example 3. *We consider one more example, which differs from the previous one. System (3) has a high dimension, and to avoid unnecessary conversions, we take $M = I$. Matrices K, N are:*

$$
K = \begin{pmatrix} 16 & -4 & 0 & 0 & -4 \\ -4 & 21 & -5 & 3 & -1 \\ 0 & -5 & 17 & -3 & 5 \\ 0 & 3 & -3 & 33 & -3 \\ -4 & -1 & 5 & -3 & 21 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 & n_{13} & n_{14} & 0 \\ 0 & 0 & 0 & n_{24} & -n_{13} \\ -n_{13} & 0 & 0 & n_{34} & 0 \\ -n_{14} & -n_{24} & -n_{34} & 0 & n_{45} \\ 0 & n_{13} & 0 & -n_{45} & 0 \end{pmatrix}.
$$

Even in this case, when matrices M, K are known, the dimension $n = 5$, and more importantly, the presence of five uncertain parameters may cause a serious obstacle for finding the stability conditions. The potential system has two pairs of multiple eigenvalues, and from a common viewpoint, it raises a query about the existence of a such matrix N (even with $\|N\| \ll 1$), which would not bring the flutter instability. Again, the direct analysis of the characteristic polynomial is not so optimistic, i.e., the algorithm described in Section 3 still works, but the expressions calculated are very bulky. In order to omit the problem occurring, we try to employ Theorem 5.

The eigenvalues and eigenvectors of matrix K are as follows:

$$
\Lambda_{1,2} = 12, \quad \beta_1 = (1,1,1,0,0)^\tau, \quad \beta_2 = (1,0,-1,0,1)^\tau, \quad \Lambda_{3,4} = 24, \quad \beta_3 = (-1,1,0,0,1)^\tau,
$$

$$
\beta_4 = (1,-2,1,1,0)^\tau, \quad \Lambda_5 = 36, \quad \beta_5 = (0,-1,1,-3,1)^\tau.
$$

As matrix $\widetilde{K} = K$ has two different multiple eigenvalues, we have to choose one of them to determine block-matrix K_1. Let this matrix correspond to $\Lambda_1 = 12$, (actually, there is no difference; the choice of $\Lambda = 24$ leads to the same conditions), then with transformation matrix T and taking into account that $q = 2$, the equality (21) of the theorem imposes the following restrictions $n_{34} = -n_{14} - n_{24}$, $n_{45} = 2n_{14} + n_{24}$, and hence, the condition (22) takes the form:

$$
T_q N T_q^\tau = n_{13} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
$$

Thereby, if $n_{13} \neq 0$, the system (MKN) is unstable (even with an infinitely small magnitude of circulatory force). However, if $n_{13} = 0$, components n_{14}, n_{24} are responsible for stability or instability. As the system is decomposed, the characteristic polynomial is reduced to third order form:

$$\lambda^3 + 7\lambda^2 + [16 + 3(n_{14} + n_{24})^2]\lambda + 12 + \frac{51}{4}n_{14}^2 + 12n_{14}n_{24} + 6n_{24}^2.$$

Its coefficients are positive, and its discriminant:

$$D = \frac{16}{3}[-4608n_{14}^6 - 13824n_{14}^5 n_{24} + (543 - 20736n_{24}^2)n_{14}^4 + (816 - 18432n_{24}^2)n_{14}^3 n_{24} -$$

$$- (16 - 600n_{24}^2 + 10368n_{24}^4)n_{14}^2 + (192 - 3456n_{24}^2)n_{14}n_{24}^3 + 48n_{24}^4 - 576n_{24}^6]$$

is positive in the domains shown in Figure 5. This result can be interpreted in the following way: when the potential system has multiple eigenvalues, the circulatory forces may contain "harsh" components, which break the stability no matter how small they are, and "soft" components are "dangerous" only if they are high enough.
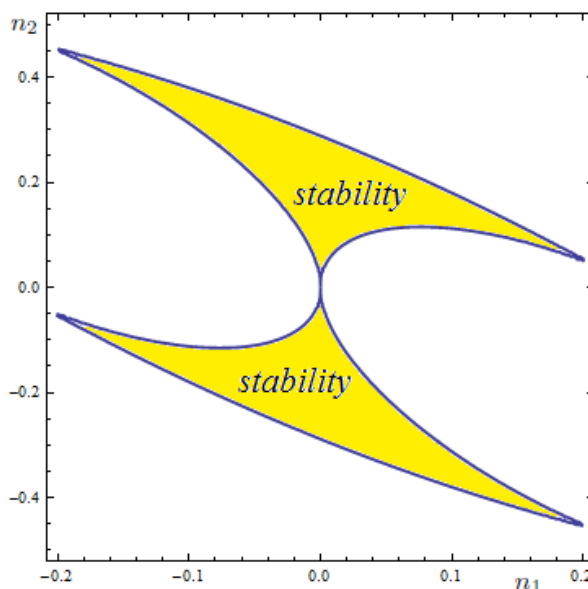

Figure 5. Stability domains for Example 3.

5. Nonlinear Systems with Unstable Equilibrium

5.1. Some General Remarks

Theorem 5 leads to the following remarkable consequence: if the linear part of the nonlinear dynamical system:

$$M\ddot{x} + (D + G)\dot{x} + (K + N)x = F(x, \dot{x})$$

is decomposed into unstable and stable (even asymptotically stable) subsystems and these subsystems are connected by nonlinear terms, then the equilibrium of the system is unstable. Moreover, the linearly stable component may bring more "discords" into nonlinear dynamics. Let us explain this by employing the following example:

$$M = I_4, \ K = \text{diag}(1, 1, 2, 3), \ N = n_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus n_2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{27}$$

$$D = 0, \ G = 0, \ F_1 = (1.2x_1 x_4, -0.5x_3^2, -n_2 x_2 x_4^2, n_2 x_1 x_3^2)^\tau.$$

The linear part of System (27) is taken from Example 2 (with $n_1 \neq 0$), so it is "semi-stable", i.e., it consists of unstable and stable subsystems, and their eigenvalues are presented in Remark 1. To be able to perform numerical integration of the equations considered, we take $n_1 = 0.1, n_2 = 0.3$, then $\lambda_{1,2,3,4} = \pm 0.0499 \pm 1.0012i, \lambda_{5,6} = \pm 1.7029i, \lambda_{7,8} = \pm 1.4491i$. If the initial perturbations are sufficiently small, then for a period of time, the motion is regular. For System (27) with initial values $x(0) = (0.001, 0.03, -0.001, 0.001)^\tau$, $\dot{x}(0) = (0.01, 0.012, 0.001, 0.02)^\tau$, this interval is approximately estimated as $[0, 110]$. The maximal Lyapunov exponent is not very big (≈ 0.05), and the behavior with respect to x_1, x_2 is similar to linear instability (flutter). However, at some point, $t_{crit} \approx 111$, the system dynamics very rapidly becomes chaotic, and in a few seconds, the trajectory whirls away (leaves the visible limits; and the computation is interrupted). The clear sign of chaotic behavior is the fact that the very small change of even one initial value ($x_1(0)$ from 0.001 to -0.001) leads to the divergence of trajectories (see Figure 6).
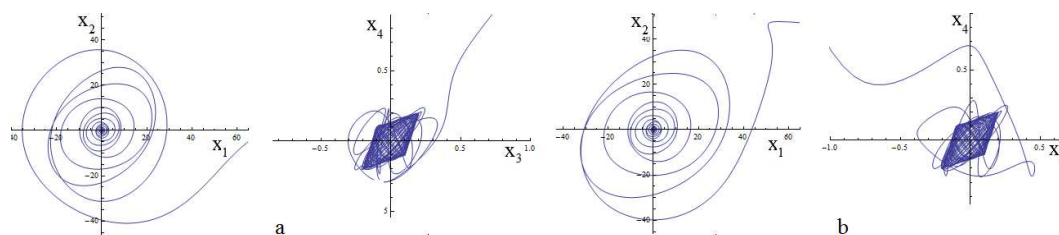

Figure 6. Projection of the phase trajectory: (a) $x_1(0) = 0.001$; (b) x_1 is changed to -0.001.

However, in applied research, Lyapunov's instability is not always a "desperate" case. Often, for reliable operation of the device, the growth of solutions is allowed in some limits. In this case, even the positive Lyapunov exponent does not necessarily lead to the failure of the system. In order to illustrate the latter statement, let us consider the following system:

$$\ddot{x}_1 + x_1(1 + x_2^2) + nx_2 = 0, \quad \ddot{x}_2 - nx_1 + x_2(1 + x_1^2) + nx_2 = 0,$$

$$\ddot{x}_3 + d\dot{x}_3(0.48 - x_1^2 - x_2^2) + 2x_3(1 - x_1^2) = 0. \tag{28}$$

Its equilibrium $x = 0$, $\dot{x} = 0$ is unstable; however, numerical simulations show that the motions are bounded, if the initial perturbations are small enough (see Figures 7 and 8). Furthermore, the behavior of component x_3 is remarkable. Contrary to Example 2, the motion is "almost" asymptotically stable with respect to this coordinate and its velocity. Speaking more strictly, it is unstable, but tends to zero as time increases. Note that time to time, there appear short intervals where the amplitude increases during two-three oscillations more then 50 times; nevertheless, it very quickly returns to small values (Figure 9).
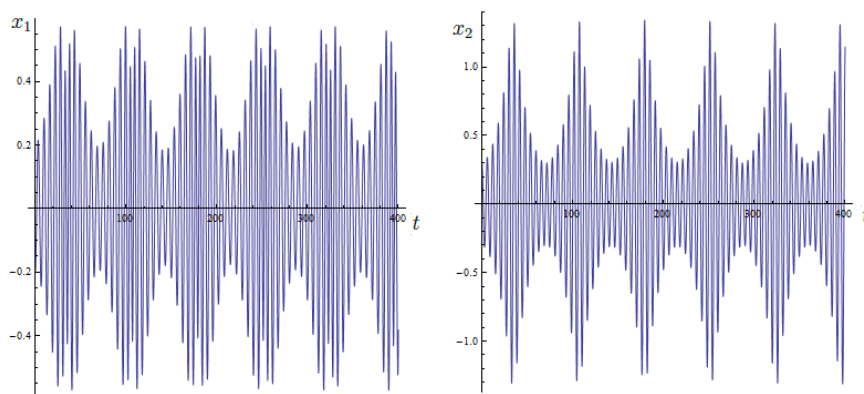

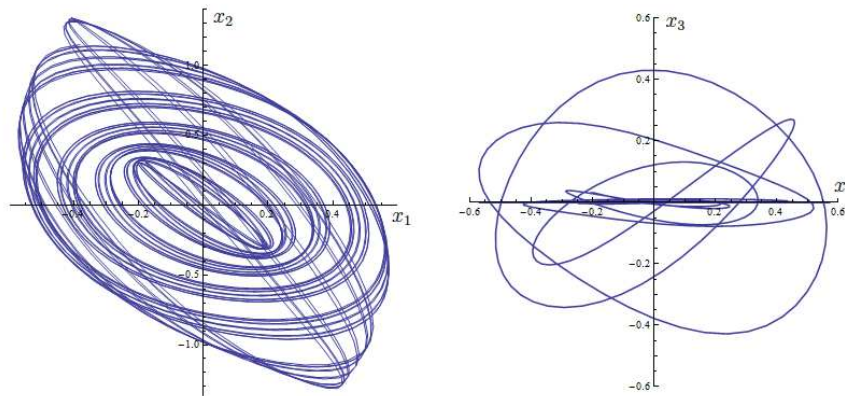**Figure 7.** Time history for "unstable" components ($d = 1.5$, $n = 0.12$).

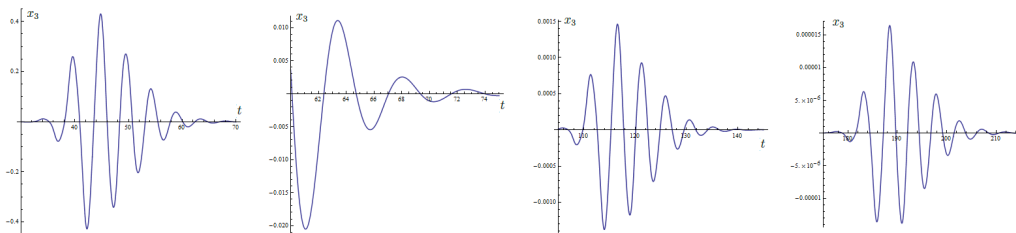**Figure 8.** Projections of the phase trajectory: boundedness of solutions ($t \in [0, 400]$).

Figure 9. Unsteady vanishing of component x_3.

5.2. A Case Study: The Bounded Solutions of a 2-DoF System with Unstable Origin

Let D be some domain of R^m containing the origin. Consider an ODE system:

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(t, \mathbf{x}), \ \mathbf{x}(t_0) = \mathbf{x}_0. \tag{29}$$

We assume that the function $\mathbf{F}$ is continuous, and the Cauchy problem for (29) has a unique solution $\mathbf{x}(t, t_0, \mathbf{x}_0)$ that continuously depends on the initial values.

For any $t \in I = [t_0, +\infty)$, we consider the domains $D_1(t), D_2(t)$ containing the origin $\mathbf{x} = \mathbf{0}$, such that their boundaries $\partial D_1, \partial D_2$ are continuous functions of time and $\partial D_l(t) \subset D_2(t)$. The following statement holds.

Lemma 1. *[39]. Assume that there exists a continuously differentiable function $U(t, \mathbf{x}) : I \times D_2 \to R$ and a continuous function $U_0(t, \mathbf{x}) \leq U(t, \mathbf{x})$ such that:*

(A) For any $t \geq t_0$, the function $U(t, \mathbf{x})$ is bounded for $\mathbf{x} \in \partial D_1(t)$;

(B) for any $(t, \mathbf{x}) \in I \times D_2(t) \backslash D_1(t)$, the time derivative of the function $U(t, \mathbf{x})$ with respect to system (1) is non-positive;

(C) for any pair $t_1, t_2 (t_2 > t_1)$ from I, the following inequality holds:

$$\sup_{\mathbf{x} \in \partial D_1(t_1)} U(t_1, \mathbf{x}) < \inf_{\mathbf{x} \in \partial D_2(t_2)} U_0(t_2, \mathbf{x}).$$

Then, for any $t_0 \in I$, the premise $\mathbf{x}_0 \in D_1(t_0)$ implies that $\mathbf{x}(t, t_0, \mathbf{x}) \in D_2(t)$.

This statement may be applied for different purposes: to prove the stability (marginal or asymptotic), the boundedness of the solutions, the estimation of the rate of the asymptotic behavior, etc.

To illustrate it, let us consider the following system:

$$\ddot{x}_1 + h\dot{x}_1 x_1^2 + x_1(k_1 - k_2 x_1^2) + nx_2 = 0, \ \ddot{x}_2 + h\dot{x}_2 x_2^2 - nx_1 + x_2(k_1 - k_2 x_2^2) = 0, \tag{30}$$

Here, h, k_1, k_2, n are some positive dimensionless parameters. System (30) may be considered as the equations of motion of two identical non-linear oscillators connected due to the influence of circulatory force. This system has a single equilibrium $x = 0, \dot{x} = 0$, which is asymptotically stable if $n = 0$ and is unstable for any arbitrary small non-zero value of n. The eigenvalues of the linearized system are located at the tops of the square in the complex plane; their values are:

$$\lambda_{1,2} = \frac{1}{\sqrt{2R}}[n \pm i(k_1 + R)], \; \lambda_{3,4} = \frac{1}{\sqrt{2R}}[-n \pm i(k_1 + R)], \; R = \sqrt{k_1 + \sqrt{k_1^2 + n^2}}. \tag{31}$$

Thus, the maximal Lyapunov exponent is equal to $n/\sqrt{2}R$, and the origin is exponentially unstable. However, if the ratio n/k_1 is small enough, then all trajectories incipient in the vicinity of the origin are bounded. To prove this fact, we shall use the lemma.

First of all, we realign the variables, parameters, and time according to formulas:

$$x_j = \sqrt{\frac{k_1}{k_2}}\tilde{x}_j \; (j = 1, 2), \; h = \frac{k_2}{\sqrt{k_1}}\tilde{h}, \; t = \frac{\tilde{t}}{\sqrt{k_1}}. \tag{32}$$

This allows formally putting the values of k_1 and k_2 in Equation (30) equal to one (for convenience, the symbol "$\sim$" is omitted below). As the system considered is autonomous, we shall seek the polynomial positive definite auxiliary function as the sum of the quadratic part and the form of fourth order: $U(x, \dot{x}) = U^{(2)} + U^{(4)}$. Obviously, it is impossible to achieve the negative definiteness of $\dot{U}^{(2)}$ (the presence of a positive Lyapunov exponent); thus, we aim for the negativeness of $\dot{U}^{(4)}$ to satisfy Condition (B) of the lemma. The function U is taken in the following form:

$$U(x_1, x_2, \dot{x}_1, \dot{x}_2) = x_1^2 + x_2^2 + \dot{x}_1^2 + \dot{x}_2^2 - \frac{1}{16}[3\dot{x}_1^4 + 6x_1^2\dot{x}_1^2 - 5x_1^4 + 3\dot{x}_2^4+ \tag{33}$$

$$+6x_2^2\dot{x}_2^2 - 5x_2^4 - 4h[\dot{x}_1 x_1(\dot{x}_1^2 + x_1^2) + \dot{x}_2 x_2(\dot{x}_2^2 + x_2^2)],$$

and its time derivative with respect to System (30) is:

$$\dot{U} = 2n(\dot{x}_1 x_2 - \dot{x}_2 x_1) - \frac{1}{4}h[(\dot{x}_1^2 + x_1^2)^2 + (\dot{x}_2^2 + x_2^2)^2 + nx_1 x_2(x_1^2 - x_2^2 - 3\dot{x}_1^2 + 3\dot{x}_2^2)]+$$

$$+ \frac{3}{4}n[\dot{x}_1 x_2(\dot{x}_1^2 + x_1^2) - \dot{x}_2 x_1(\dot{x}_2^2 + x_2^2)] - \frac{1}{4}[hx_1^2(3\dot{x}_1^4 + 6\dot{x}_1^2 x_1^2 - x_1^4)+ \tag{34}$$

$$+hx_2^2(3\dot{x}_2^4 + 6\dot{x}_2^2 x_2^2 - x_2^4) + (3 - h^2)(\dot{x}_1 x_1^5 + \dot{x}_2 x_2^5) + (3 + h^2)(\dot{x}_1^3 x_1^3 + \dot{x}_2^3 x_2^3)]$$

The upper bound for derivative $\dot{U}(x, \dot{x})$ may be derived in the form:

$$\dot{U} \leq \tilde{U}(r_1, r_2) \triangleq -\frac{1}{4}\{-nr_1 r_2(4 + r_1^2 + r_2^2) + h(r_1^4 + r_2^4)- \tag{35}$$

$$-\frac{3}{4}np_1(r_1^2 + r_2^2)^2 - [4h + \sqrt{h^4 + 21h^2 + 36} + p_2](r_1^6 + r_2^6)\}.$$

Here, $r_j = \sqrt{x_j^2 + \dot{x}_j^2} \; (j = 1, 2)$, $p_1 \approx 1.4$, and the value of $p_2(h)$ depends on parameter h and is defined as:

$$p_2(h) = h \max_{\xi \in R} \frac{\xi^6 - 15\xi^4 + 15\xi^2 - 1 + 2h\xi(3\xi^4 - 10\xi^2 + 3)}{(\xi^2 + 1)^3}. \tag{36}$$
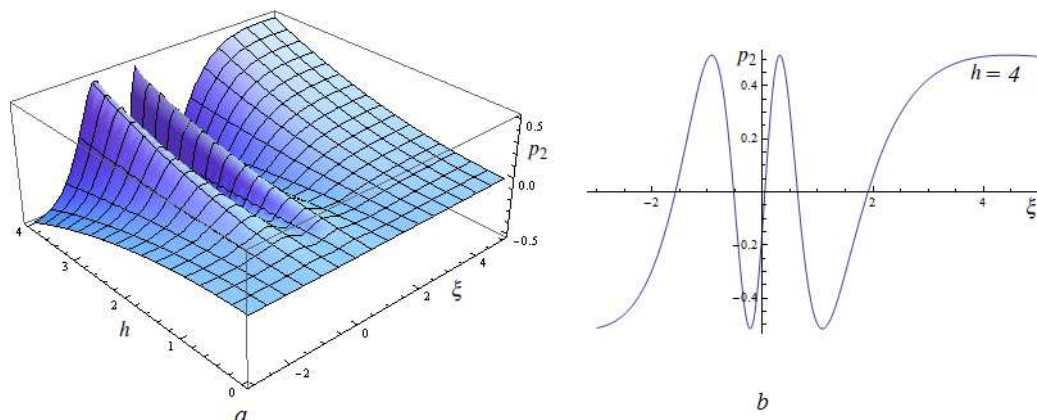
The surface $p = p_2(h, \xi)$ is presented in Figure 10.

Figure 10. The view of the surface $p = p_2(h, \xi)$.

From the other side, the function U satisfies the following inequalities:

$$U_0 \triangleq r_1^2 + r_2^2 - \frac{1}{16}(\sqrt{h^2 + 1} + 4)(r_1^4 + r_2^4) \leq U(x, \dot{x}) \leq r_1^2 + r_2^2$$

$$- \frac{1}{16}(\sqrt{h^2 + 1} + 4)(r_1^4 + r_2^4) \triangleq U_1. \tag{37}$$

In the vicinity of the origin of the first quadrant of the plane $Or_1 r_2$, the boundary of the set $\widetilde{U}(r_1, r_2) = 0$ is formed by two branches (solid lines in Figure 11a). The smaller one bounds the domain $\widetilde{U} > 0$. Thus, we should choose the domain D_1 in such a way that ∂D_1 belongs to domain $\widetilde{U} \leq 0$. It can be taken as $U_1(r_1, r_2) \leq \alpha(h, n)$, where α is some constant number. Then, the boundary of D_2 may be given by condition $U_0 = \alpha + \varepsilon$, where ε is an arbitrary small positive number. Therefore, all conditions of the lemma are fulfilled, and all trajectories starting in D_1 are limited by the threshold of ∂D_2. For values $h = 4, n = 0.1$, these domains are presented in Figure 11.

This conclusion is confirmed by numerical integration of System (30). For different starting points of the phase space, the trajectories go to the limit cycle (Figure 12). In fact, the estimation shown in Figure 11 is accurate enough.
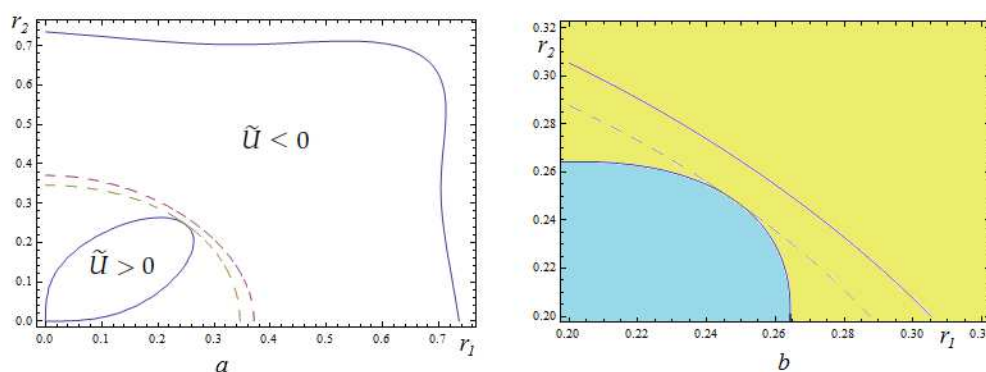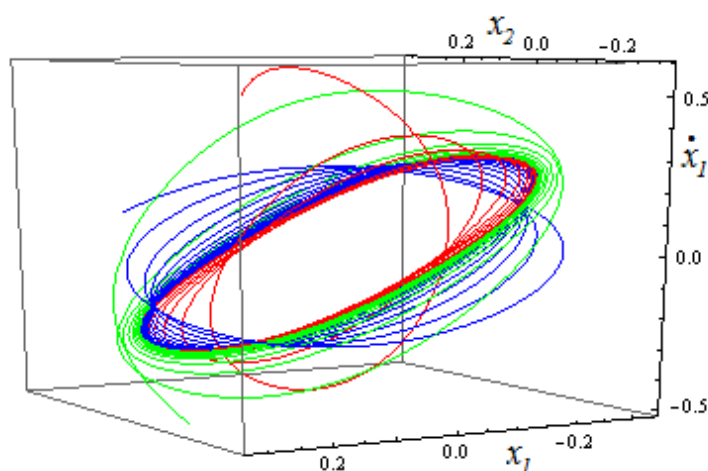

Figure 11. The domains where all trajectories starting in D_1 are limited by the threshold of ∂D_2.

Figure 12. The limit cycle of the trajectories for different starting points of the phase space.

6. Concluding Remarks

In this paper, we discussed some stability problems for circulatory dynamical systems. Although existing analytical methods allow us to solve the problem (obtaining stability conditions in the closed form) in the case of high order systems with uncertain parameters, these conditions may turn out to be too cumbersome for analysis. Therefore, it is advisable to use optimized algorithms from the point of view of avoiding "unnecessary" computational procedures, which allows reducing the computation time and, possibly, obtaining stability conditions in a simpler form. Some of these procedures were discussed in Section 2 and tested in Section 3. In Section 4, a case of multiple eigenvalues was studied. It is interesting from a theoretical viewpoint as well, as this case often occurs in different applications. A sufficient condition of instability was formulated. In Section 5, some interesting examples of nonlinear dynamics were considered. Then, it was demonstrated that the flutter instability of the linearized system may lead to various results, namely the motion may become chaotic with overwhelming growth of the amplitude and may be bounded. Actually, it may be even asymptotically stable with respect to part of the variables. However, the rigorous study of this question is based on the use of Lyapunov functions for a nonlinear multi-parameter system, which leads to extremely complicated transformations and requires a separate study.

The future research will consider the problem of free flexural vibrations and dynamic loads in an elastic rod with stepwise-changing stiffness, as well as the application of the stated results to the hemivariational principles such as damage [40,41].

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Appendix A

Formulas for the G-approach (Section 2), $n = 4$:

$$s_1 = -\frac{a_3}{a_4}, \ s_2 = \frac{1}{a_4^2}, \ s_3 = -\frac{1}{a_4^3}(a_3^3 - 3a_2a_3a_4 + 3a_1a_4^2), \ s_4 = \frac{1}{a_4^4}(a_3^4 - 4a_2a_3^2a_4 +$$

$$+4a_1a_3a_4^2 + 2a_2^2a_4^2 - 4a_0a_4^3),\ s_5 = -\frac{1}{a_4^5}(a_3^5 - 5a_2a_3^3a_4 + 5a_1a_3^2a_4^2 + 5a_2^2a_3a_4^2 - 5a_0a_3a_4^3-$$

$$-5a_1a_2a_4^3),\ s_6 = \frac{1}{a_4^6}(a_3^6 - 6a_2a_3^4a_4 + 6a_1a_3^3a_4^2 + 9a_2^2a_3^2a_4^2 - 6a_0a_3^2a_4^3 - 12a_1a_2a_3a_4^3-$$

$$2a_2^3a_4^3 + 6a_0a_2a_4^4 + 3a_1^2a_4^4),\ \Delta_2 = \frac{1}{a_4^2}(3a_3^2 - 8a_2a_4),\ \Delta_3 = \frac{2}{a_4^4}(-3a_1a_3^3 + a_2^2a_3^2$$

$$-6a_0a_3^2a_4 + 14a_1a_2a_3a_4 - 4a_2^3a_4 + 16a_0a_2a_4^2 - 18a_1^2a_4^2),$$

$$\Delta_4 = \frac{1}{a_4^6}(-192a_0^2a_1a_3a_4^2 - 4a_0a_2^3a_3^2 + a_1^2a_2^2a_3^2 + 144a_0^2a_2a_3^2a_4 - 6a_0a_1^2a_3^2a_4+$$

$$+18a_1^3a_2a_3a_4 + 144a_0a_1^2a_2a_4^2 - 4a_1^3a_3^3 - 27a_0^2a_3^4 + 16a_0a_2^4a_4 - 4a_1^2a_2^3a_4 - 128a_0^2a_2^2a_4^2+$$

$$+18a_0a_1a_2a_3^3 - 80a_0a_1a_2^2a_3a_4 + 256a_0^3a_4^3 - 27a_1^4a_4^2.$$

The discriminant for Example 1:

$$D(f) = 4\beta^6 - 192\beta^5(\gamma_1 + 6\gamma_2 + 8) + 4\beta^4(25\gamma_1^2 + 366\gamma_1\gamma_2 + 945\gamma_2^2 + 406\gamma_1 + 2418\gamma_2+$$

$$+1801) - 16\beta^3(3\gamma_1^3 + 47\gamma_1^2\gamma_2 + 261\gamma_1\gamma_2^2 + 369\gamma_2^3 + 25\gamma_1^2 + 554\gamma_1\gamma_2 + 1629\gamma_2^2 + 421\gamma_1+$$

$$+2067\gamma_2 + 1135) + 8\beta^2(2\gamma_1^4 + 27\gamma_1^3\gamma_2 + 192\gamma_1^2\gamma_2^2 + 579\gamma_1\gamma_2^3 + 450\gamma_2^4 + 18\gamma_1^3 + 437\gamma_1^2\gamma_2+$$

$$+2640\gamma_1\gamma_2^2 + +4641\gamma_2^3 + 32\gamma_1^2 + 2151\gamma_1\gamma_2 + 7016\gamma_2^2 + 2322\gamma_1 + 8473\gamma_2 + 2450)-$$

$$-16\beta[\gamma_1\gamma_2(\gamma_1^3 + 9\gamma_1^2\gamma_2 + 47\gamma_1\gamma_2^2 + 75\gamma_2^3) + 6\gamma_1^4 + 62\gamma_1^3\gamma_2 + 415\gamma_1^2\gamma_2^2 + 1196\gamma_1\gamma_2^3+$$

$$+1425\gamma_2^4 - 30\gamma_1^3 + 156\gamma_1^2\gamma_2 + 1541\gamma_1\gamma_2^2 + 3477\gamma_2^3 + 138\gamma_1^2 + 1294\gamma_1\gamma_2 + 3667\gamma_2^2+$$

$$+1470\gamma_1 + 4655\gamma_2] + 4\gamma_1^2\gamma_2^2(\gamma_1^2 + 6\gamma_1\gamma_2 + 25\gamma_2^2) + 2\gamma_1\gamma_2(2\gamma_1^3 + 8\gamma_1^2\gamma_2 + 51\gamma_1\gamma_2^2 - 25\gamma_2^3)+$$

$$+36\gamma_1^4 + 372\gamma_1^3\gamma_2 + +2454\gamma_1^2\gamma_2^2 + 6858\gamma_1\gamma_2^3 + 9025\gamma_2^4 - 360\gamma_1^3 - 1572\gamma_1^2\gamma_2 - 4208\gamma_1\gamma_2^2-$$

$$-2166\gamma_2^3 + 1764\gamma_1^+ 28428\gamma_1\gamma_2 + 17689\gamma_2^2.$$

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