Journal of Sound and Vibration 523 (2022) 116699

Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsvi

On the stability of the equilibrium of the double pendulum with follower force: Some new results

Volodymyr Puzyrov [a,c], Jan Awrejcewicz [b,*], Nataliya Losyeva [c], Nina Savchenko [d]

[a] Universitat Politecnica de Catalunya, Terrassa, Spain
[b] Lodz University of Technology, Lodz, Poland
[c] Nizhyn Gogol State University, Nizhyn, Ukraine
[d] Zhukovsky National Aerospace University "KhAI", Kharkiv, Ukraine

ARTICLE INFO

Keywords:
Stability
Weak damping
Non-conservative forces
Critical load
Flutter zone

ABSTRACT

The article deals with the problem of the stability of the equilibrium of a double pendulum under the action of a follower force. The latter is assumed to be imperfect, that is, the direction of its action does not coincide with the axis of the second link of the pendulum. The relationships between the masses of the pendulums, the stiffnesses of the hinges, and the damping coefficients are assumed to be arbitrary. Conditions of stability in the absence of damping and for a weakly damped system are obtained and analyzed. These conditions are formulated for the linearized motion equations in the form of an estimate of the dimensionless parameter corresponding to the value of the critical load. The influence of damping ratio and stiffness ratio on the permissible value of this parameter has been studied. It is shown that near the boundary of the region of flutter instability, the value of the critical load can be increased by decreasing the stiffness of the hinge at the attachment point of the pendulum.

1. Introduction

The history of research on the stability of columns subjected to compressive forces dates back to Euler, who analyzed the static buckling of elastic compressed rods and formulated a theory of buckling, which can be regarded as the first approximation in solving the stability problem of lightweight load-bearing structures. E. L. Nikolai was, in all likelihood, the first who, in the first half of the last century, considered the problem of the stability of an elastic system loaded with the so-called follower forces, which change their direction in accordance with the current configuration of the system on which they act. The problem was updated and attracted much attention of scientists in the field of structural mechanics in the second half of the last century, when flutter was theoretically discovered as a result of the action of such forces (Bolotin [1]).

In 1952, Hans Ziegler published an article [2], which is now considered a classic and widely known in the community of mechanical engineers and specialists in dynamics. Ziegler investigated the flutter problem in aerodynamics and considered the model as a double pendulum fixed at one end and loaded with a tangential load at the other end. He discovered a phenomenon with an unexpected property: there is a gap between the stability regions in the case of vanishingly small damping compared to the case when there is no damping. (This effect is a classic example of Hopf bifurcation in a dynamical system.) This problem attracted a lot of attention of scientists in the field of structural mechanics in the second half of the last century, and numerous studies were devoted to the theoretical analysis of the flutter arising from the action of follower forces [3–21].

* Corresponding author.
E-mail address: jan.awrejcewicz@p.lodz.pl (J. Awrejcewicz).

https://doi.org/10.1016/j.jsv.2021.116699
Received 11 October 2021; Received in revised form 30 November 2021; Accepted 13 December 2021
Available online 7 January 2022

Koval'chuk and Lobas [22] analyzed the influence of the orientation parameter of the behavior of the pendulum including a simple zero eigenvalue in the linearization matrix. Lobas [17] derived differential equation governing the plane-parallel motion of an inverted multilink pendulum with an asymmetric follower force acting at the elastically restrained upper end. The nonlinear springs and dissipation feature were considered. Lobas and Koval'chuk [23] investigated stability of the vertical pendulum equilibrium with both linear and nonlinear elastic elements in the critical case. Lobas and Ichanskii [16] studied two limit cycles one stable and one unstable of a double pendulum under a follower force action. Their study was extended to include analysis of the limit cycles of a double pendulum with linear and nonlinear springs subjected to a follower force based on the numerical simulations.

In addition, the problem of destabilizing the equilibrium of a non-conservative mechanical system under the action of low friction was reflected in many applied problems. Among them are: vibrations caused by friction [24–27], squealing vibrations in drum brakes [28], the Levitron system [29], aero-elastic systems [30], the influence of rocket thrust as a source of tracking load acting on a beam [31] and others.

A significant part of the research is devoted to the study of the Ziegler model, its various generalizations and modifications. As the main results of the last decade, we note the following articles.

The article [32] concerns the destabilization paradox occurring in damped circulatory systems (both discrete and continuous). The concepts of "sensitivity to damping" and "degree of nonconservativeness" have been discussed by referring to the generalized Beck problem.

In paper [33] authors presented some new results using different methods of proof on the influence of the damping terms which may be applied for avoidance of self-excited vibrations in circulatory systems.

The work [34] studies the stability of so-called **MDGKN** system (i.e. the linear system where matrices **M**, **D**, **G**, **K**, **N** are associated with inertia, damping, gyroscopic, potential and circulatory forces respectively) with two degrees of freedom with regard to infinitesimally small, incomplete, and indefinite damping matrices as well as the role of gyroscopic terms and the spacing of the eigenfrequencies.

Luongo et al. [35] studied the destabilizing effect of damping on both linear and nonlinear behavior of the Ziegler column. An algorithm based on the Multiple Scale Method is developed to investigate the post-critical behavior of the system.

Abdullatif et al. [3] discussed the phenomenon of three damping-induced stability transitions in non-conservative systems. It is shown that the addition of damping can cause non-conservative systems to become stable, then unstable, then stable again at the same value of the non-conservative forcing variable.

Kirillow and Bigoni [36] presented an overview of results and methods of stability theory that are specific for nonconservative applications. Various topics are discussed: flutter and divergence, reversible- and Hamiltonian–Hopf bifurcation, dissipation-induced instabilities, destabilization paradox, influence of structure of forces on stability and others.

D'Annibale and Feretti [37] discussed the effects of linear damping on the post-critical behavior of the Ziegler's column. The analysis of nonlinear behavior of a generically damped column based on the Multiple Scale Method is presented. Asymptotic results obtained have been validated with numerical solutions of the exact equations of motion.

The aim of our study is to obtain and analyze stability conditions for equilibria of the double pendulum with imperfect (non-tangential) follower force for uncertain values of mass, damping and stiffness ratios. The cases of undamped and weakly-damped systems are considered. The possibility of increasing the value of critical load by varying the damping and stiffness ratios is discussed. In particular, it is shown that in the vicinity of the boundary of flutter instability zone in parametric space this value can be increased by softening the spring at the base of the pendulum.

In the present paper we analyze the stability conditions for linearized model. The main focus is on assessing the influence of system parameters (mass, damping and stiffness ratios and asymmetry index) onto the value of critical load.

The article is organized in a following way. In Section 2 the problem statement is formulated and some preliminary simplifications are given. In Section 3, the conditions for the stability of the system in the absence of damping in hinges of the pendulum are obtained and analyzed. In Section 4, the case of weak damping is considered and conditions for the asymptotic stability of the equilibrium under study are found. Section 5 discusses the influence of damping ratio and stiffness ratio on the value of critical load, and presents the results of numerical investigations.

2. Statement of the problem

Consider the double pendulum shown in Fig. 1, consisting of two rigid weightless rods of equal length L, carrying the concentrated masses m_1 and m_2. The rods are connected by a viscoelastic joint, and the configuration of the system is determined by the two angles φ_1 and φ_2 formed between the vertical axis and each of the two rods, respectively. The follower force is applied to the free end. (The case $m_1 = 2m_2, c_1 = c_2$ corresponds to Ziegler's standard model [2].)

The kinetic energy and potential energy of the system have the following forms

$$K = \frac{1}{2} l^2 [(m_1 + m_2)\dot{\varphi}_1^2 + 2m_2\dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_2 - \varphi_1) + m_2\dot{\varphi}_2^2], \tag{1}$$

$$\Pi = \frac{1}{2}[k_1\varphi_1^2 + k_2(\varphi_1 - \varphi_2)^2], \tag{2}$$

and the non-conservative forces are

$$Q_1 = -(\widetilde{d}_1 + \widetilde{d}_2)\dot{\varphi}_1 + \widetilde{d}_2\dot{\varphi}_+ Pl[\alpha\sin\varphi_1 + (1-\alpha)\sin(\varphi_1 - \varphi_2)],\ Q_2 = \widetilde{d}_2(\dot{\varphi}_1 - \dot{\varphi}_1) + Pl\alpha\sin\varphi_2. \tag{3}$$

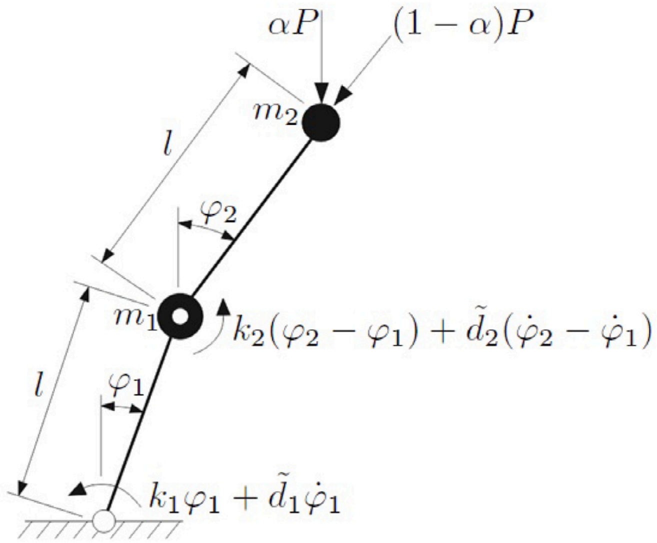

Fig. 1. The double pendulum with follower force.

The equations of the motion may be presented in Lagrange form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}_j}-\frac{\partial L}{\partial \varphi_j}=Q_j,\ \ L=K-\Pi,\ \ j=1,2. \tag{4}$$

Introducing the dimensionless parameters and time according to formulas

$$\mu=\frac{m_1}{m_2},\ d_j=\frac{\widetilde{d}_j}{l}\sqrt{\frac{1}{m_2k_2}}\ (j=1,2),\ d=\frac{d1}{d_2},\ \kappa=\frac{k_1}{k_2},\ p=\frac{Pl}{k_2},\ \tau=\sqrt{\frac{k_2}{m_2l^2}}t, \tag{5}$$

we derive the following counterpart nondimensional equations

$$\mathbf{M}\boldsymbol{\varphi}''+\mathbf{D}\boldsymbol{\varphi}'+\mathbf{K}\boldsymbol{\varphi}=\mathbf{F}(\boldsymbol{\varphi}). \tag{6}$$

Here

$$\mathbf{M}=\begin{pmatrix}1+\mu & 1\\ 1 & 1\end{pmatrix},\quad \mathbf{D}=d_2\begin{pmatrix}1+d & -1\\ -1 & 1\end{pmatrix},\quad \mathbf{K}=\begin{pmatrix}1+\kappa-p & -1+p(1-\alpha)\\ -1 & 1-p\alpha\end{pmatrix}. \tag{7}$$

and $\mathbf{F}(\boldsymbol{\varphi})$ is a set of nonlinear expansion terms.

Linearizing equations (6) in the vicinity of equilibrium $\boldsymbol{\varphi}=0, \dot{\boldsymbol{\varphi}}=\mathbf{0}$, the following characteristic polynomial is obtained

$$f_0(\lambda)=a_4\lambda^4+a_3\lambda^3+a_2\lambda^2+a_1\lambda_1+a_0, \tag{8}$$

where

$$\begin{aligned}&a_4=\mu, a_3=d_2(4+d+\mu), a_2=\kappa+4+\mu+dd_2^2-p(2+\mu\alpha),\\ &a_1=d_2[\kappa+d-\alpha p(d+1)], a_0=\alpha p^2-\alpha p(\kappa+2)+\kappa.\end{aligned} \tag{9}$$

3. The stability conditions of the linearized system (no damping)

Firstly, let us consider the undamped case which formally corresponds to the case $d_2=0$. The zero solution of the **MDK** system (we will refer it as $(6)_{lin}$) is stable (marginally) when polynomial has two pairs of purely imaginary roots which means that the following conditions

$$a_2^{(0)}(p)=a_2|_{d_2=0}>0,\ \ a_0(p)>0,\ \ H(p)=(a_2^{(0)})^2-4\mu a_0^{(0)}>0 \tag{10}$$

hold. Here the expression for H is determined by formula

$$H(p)=(\mu^2\alpha^2+4)p^2-2[\mu(\mu-\kappa)\alpha+2(\mu+\kappa+4)]p+\kappa^2+2(4-\mu)\kappa+(\mu+4)^2. \tag{11}$$

Assuming that the values of the parameters μ,κ are known, we shall seek a solution to the system of inequalities (10) depending on the value of the parameter α. Denote the discriminants of the quadratic polynomials $a_0(p), H(p)$ as dis_{a_0} and dis_H, respectively, we get

$$dis_{a_0}(\alpha)=(\kappa+2)^2\alpha^2-4\kappa\alpha,\ \ dis_H(\alpha)=-16\mu[2\mu(\mu+\kappa+2)\alpha^2+(\kappa-\mu)(\kappa+\mu+4)\alpha-4\mu\kappa], \tag{12}$$

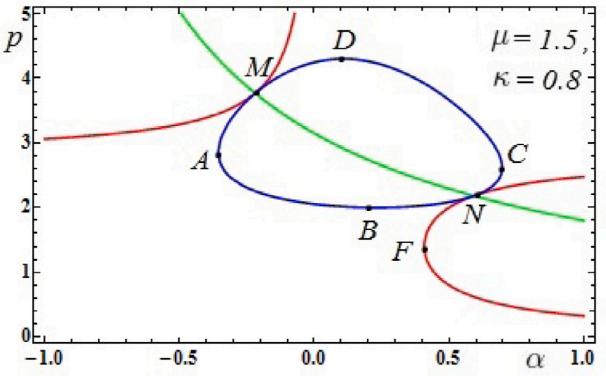

Fig. 2. Parameter family of the curves in plane αp.

The first of these expressions has a zero root $\alpha_0 = 0$ and a positive root

$$\alpha_\star = \frac{4\kappa}{(\kappa+2)^2}. \tag{13}$$

The polynomial $dis_H(\alpha)$ has two real roots of opposite signs

$$\alpha_j = \frac{1}{4\mu(\mu+\kappa+2)}[(\mu-\kappa)(\mu+\kappa+4)+(-1)^j(\mu+\kappa)\sqrt{(\kappa-\mu+4)^2+16\mu}],\ j=1,2. \tag{14}$$

Notice that for any admissible values of the parameters μ, κ, α at least one of the polynomials $a_0(p)$, $H_0(p)$ has real roots. Indeed, suppose the opposite, that is, the discriminants of these quadratic polynomials are negative. Since $\alpha_1 < \alpha_0 = 0$, this is only possible if $0 < \alpha < \alpha_\star(\mu,\kappa)$, $\alpha > \alpha_2(\mu,\kappa)$. However, it is easy to see that this requirement is impracticable. Assuredly, calculating $dis_H(\alpha_\star)$, we obtain

$$\frac{64\mu\kappa}{(\kappa+2)^4}[(2-\mu)\kappa+2(2+\mu)]^2 \geq 0, \tag{15}$$

that is the value $\alpha_\star$ does not exceed the value of α_2. Moreover, the equal sign is possible only under the condition $\mu > 2$, $\kappa = 2(\mu+2)/(\mu-2)$. In other words, in the region of negativity of the discriminant dis_{a_0}, the expression for dis_H is positive.

Taking into account that for any admissible values of the parameters μ, κ, the following inequalities hold

$$\alpha_1 < \alpha_0 < \alpha_\star \leq \alpha_2, \tag{16}$$

we shall solve the system of inequalities (10) depending on the value of the parameter α.

Geometrically, each of the equations $a_2^{(0)}(p) = 0$, $a_0(p) = 0$, $H_0(p) = 0$ defines a two-parameter family of curves in the plane αp (Fig. 2).

Their explicit equations are

$$p = p_2(\alpha) = \frac{\mu+\kappa+4}{\mu\alpha+2},\ p = p_{0j}(\alpha) = 1+\frac{\kappa}{2}+(-1)^j\frac{\sqrt{dis_{a_0}}}{\alpha}, \tag{17}$$

$$p = p_{Hj} = \frac{1}{\mu^2\alpha^2+4}[\mu(\mu-\kappa)\alpha-2(\mu+\kappa+4)+\frac{(-1)^j}{2}\sqrt{dis_H}],\ j=1,2. \tag{18}$$

Let us calculate the partial derivatives

$$\frac{\partial H}{\partial \alpha} = 2\mu p(\mu\alpha p+\kappa-\mu),\ \frac{\partial H}{\partial p} = 2[(\mu^2\alpha^2)p+\mu(\mu-\kappa)\alpha-4(\mu+\kappa+4)]. \tag{19}$$

It is easy to see that curve $H(\alpha,p)=0$ does not intersect the abscissa axis, because $H(\alpha,0) = (\mu-\kappa)^2+8(\mu+\kappa+2) \neq 0$. Thus, $\partial H/\partial\alpha = 0$ if and only if

$$\alpha = \frac{\mu-\kappa}{\mu p}. \tag{20}$$

In this case, the assumption $\partial H/\partial p = 0$ leads to the consequences

$$p = 2+\frac{\mu+\kappa}{2},\ H = -(\mu+\kappa)^2 \neq 0, \tag{21}$$

and is false for any values of parameters μ, κ. Thus, the curves $H(\alpha,p)=0$ are regular. Each curve consists of two continuous branches $p = p_{H1}(\alpha)$, $p = p_{H2}(\alpha)$. The lower and upper branches each have one stationary point, respectively

$$B(\frac{\mu-\kappa}{2\mu},2),\ D(\frac{\mu-\kappa}{\mu(2+\mu+\kappa)},\mu+\kappa+2). \tag{22}$$

We calculate the value of the second derivative $\frac{d^2p}{d\alpha^2}$ at these points. Using the well-known formula

$$\frac{d^2p}{d\alpha^2} = -\frac{1}{(\partial H/\partial p)^3}[H_{\alpha\alpha}(\frac{\partial H}{\partial p})^2 - 2H_{\alpha p}\frac{\partial H}{\partial \alpha}\frac{\partial H}{\partial p} + H_{pp}(\frac{\partial H}{\partial \alpha})^2], \tag{23}$$

we obtain

$$\frac{d^2p}{d\alpha^2}(\alpha_B, p_B) = \frac{2\mu^2}{\mu+\kappa} > 0, \ \frac{d^2p}{d\alpha^2}(\alpha_D, p_D) = -\frac{\mu^2(\mu+\kappa+2)^2}{2(\mu+\kappa)} < 0. \tag{24}$$

As a consequence, the function $p = p_{H1}(\alpha)$ decreases on the interval $[\alpha_A, \alpha_B)$ and increases as $\alpha \in (\alpha_B, \alpha_C]$; the function $p = p_{H2}(\alpha)$ increases on the interval $[\alpha_A, \alpha_D)$ and decreases as $\alpha \in (\alpha_D, \alpha_C]$.

Thus, geometrically, the condition $H = 0$ may be considered as a two-parameter family of closed regular curves in the plane αp with apexes at the points A (left apex), B (downside apex), C (right apex), D (upside apex).

Below we will repeatedly use the well-known equivalence of inequalities, which we formulate in the form of the following three statements.

Let $\psi(\xi) = b_2\xi^2 + b_1\xi + b_0$ be the real-valued quadratic polynomial with two different real roots $\xi_1 < \xi_2$, and ξ_0 is some arbitrary number.

Statement 1. The inequality $\xi_0 < \xi_1$ is equivalent to system of inequalities

$$b_2\psi(\xi_0) > 0, \ \xi_0 < -\frac{b_1}{2b_2}. \tag{25}$$

Statement 2. The double inequality $\xi_1 < \xi_0 < \xi_2$ is equivalent to inequality $b_2\psi(\xi_0) < 0$.

Statement 3. The inequality $\xi_0 > \xi_2$ is equivalent to system of inequalities

$$b_2\psi(\xi_0) > 0, \ \xi_0 > -\frac{b_1}{2b_2}. \tag{26}$$

The following cases are possible depending on the value of the parameter α.

(A1) $\alpha < \alpha_1$. In this case, the inequality $dis_H(\alpha, \kappa, \mu) < 0$ takes place, that is, H_0 is positive for any value of the parameter p. To find the stability condition, it is necessary to compare the expressions $p_2(\alpha)$ and $p_{0j}(\alpha)$.

Since $\alpha < 0$, then the condition $a_0(p) > 0$ is equivalent to the inequality $p < p_{01}(\alpha, \mu, \kappa)$. In other words, for any positive value of p from $a_0(p) < 0$ follows $p > p_{01}$. Calculate the value of $a_0(p_2)$

$$a_0(p_2) = -\frac{1}{(\mu\alpha+2)^2}[\mu(\kappa^2+6\kappa+2\mu+8)\alpha^2 + (\kappa^2+4\kappa-4\mu\kappa-\mu^2-4\mu)\alpha - 4\kappa]. \tag{27}$$

The expression in square brackets admits the following representation

$$[\cdots] = -\frac{dis_H}{16\mu} + \mu[(\kappa+2)^2\alpha^2 - 4\kappa\alpha]. \tag{28}$$

Since in the case under consideration we have $\alpha < \alpha_1 < \alpha_2$, then all terms on the right-hand side of equality (28) are positive. Thus, taking into account that $b_2 = \alpha < 0$, according to Statement 2 we conclude that $p_2 > p_{01}$ (the curve $p = p_2(\alpha)$ passes above the upper left branch of the curve $a_0(p) = 0$). Hence, the inequalities (10) are fulfilled if and only if $p < p_{01}(\alpha)$.

(A2) $\alpha_1 \le \alpha < 0$. The stability conditions (10) are satisfied either under the conditions $p < p_{01}, \ p < p_{H1}, p < p_2$, or in the case $p < p_{01}, \ p > p_{H2}, \ p < p_2$.

Let us first consider the possibility of intersection of the curves from the families $p = p_{01}$ and $H(p) = 0$. For this purpose, we write down the resultant of the polynomials $a_0(p)$ and $H(p)$. After some manipulations, this expression may be presented in the following form

$$rez(a_0, H) = [\mu(\kappa^2+6\kappa+2\mu+8)\alpha^2 + (\kappa^2+4\kappa-4\mu\kappa-\mu^2-4\mu)\alpha - 4\kappa]^2. \tag{29}$$

The expression in square brackets considered as polynomial with respect to parameter α has two real roots (one is positive and other is negative), which correspond to the common points of the curves under consideration.[1] Since these roots are multiple, then they are the roots of the derivative $\partial rez/\partial\alpha$, which, according to the well-known theorem from differential geometry, means tangency of the curves $a_0(p, \alpha) = 0$ and $H_0(p, \alpha) = 0$ at points M, N of the plane $\alpha - p$.

Lemma. *In this case study the inequality $p_{H2}(\alpha, \mu, \kappa) \le p_{01}(\alpha, \mu, \kappa)$ holds (the equality, as it was established above, takes place only at point M).*

Proof. We use Statement 3, where as $\psi(\xi)$ we take $H(p)$, and $\xi_0 = p_{01}$. We have the following expression

$$H(p)_{p=p_{01}} = \psi_{10}(\alpha, \mu, \kappa) - \psi_{11}(\alpha, \mu, \kappa)\sqrt{dis_0}, \tag{30}$$

where

$$\psi_{10} = 2\alpha\{\mu^2(\kappa+2)^2\alpha^3 + 2\mu[\kappa^2 - 2\kappa(\mu-1) - 2\mu]\alpha^2 + 2[\kappa^2 - 4\kappa(\mu-1) + \mu^2 + 4\mu + 8]\alpha - 8\kappa\}, \tag{31}$$

[1] In the case under consideration, we are interested in the negative value of α.

$$\psi_{11} = 2\alpha(2+\mu\alpha)[\mu\alpha(2+\kappa) - 2(\mu+2)]. \tag{32}$$

The expression (30) after identical manipulations may be presented in the following form

$$\psi_{10} = \alpha\{2\mu^2(\kappa+2)\alpha^3 - 4\mu^2\alpha^2 + [\kappa^2 + 6\kappa + 2(\mu^2+4\mu+8)]\alpha - 4\kappa + \kappa(\mu\alpha+1)^2[(\kappa+2)\alpha - 4]\}. \tag{33}$$

All parts of the sum in braces are negative, and, as well as $\alpha < 0$ then ψ_{10} is positive.

Taking into account that $\psi_{10}^2 - \psi_{11}^2 dis_0 = 16\alpha^2 rez(a_0, H)$ and, according to formula (29), $rez(a_0, H)$ is positive (with the exception of point M), we can conclude that the inequality

$$\psi_{10}(\alpha,\mu,\kappa) > |\psi_{11}(\alpha,\mu,\kappa)|\sqrt{dis_0} \tag{34}$$

holds, i.e. the first inequality of the Statement 3 is fulfilled.

Now we have to compare p_{01} and $-b_1/2b_2$ which in this case reads as expression

$$[\mu(\mu-\kappa)\alpha + 2(\mu+\kappa+4)]/(4+\mu^2\alpha^2). \tag{35}$$

Their difference taken with the positive multiplier $2(4+\mu^2\alpha^2)$ has the following form

$$-\alpha(4+\mu^2\alpha^2)\sqrt{dis_0} + [\mu^2(\kappa+2)\alpha^2 - 4\mu(\kappa-\mu)\alpha + 4(3\kappa+2\mu+10)]. \tag{36}$$

Let us show that the expression in square brackets is positive in the case under consideration. Suppose the opposite, which is equivalent to the inequality

$$\kappa \le \tilde{\kappa} = -\frac{2(\mu^2\alpha^2 + 2\mu^2\alpha + 4\mu + 20)}{\mu^2\alpha^2 - 4\mu\alpha + 12}. \tag{37}$$

We use Statement 2 now, here we take dis_H as $\psi(\xi)$, considered as a quadratic polynomial in κ, and as ξ_0 we take the expression on the right-hand side of inequality (36). As a result, we have

$$dis_H|_{\kappa=\tilde{\kappa}} = -\frac{16\mu(4+\mu^2\alpha^2)}{(\mu^2\alpha^2 - 4\mu\alpha + 12)^2}[2\mu^4\alpha^2 - \mu^2\alpha^3(\mu^2+28\mu+20) + 8\mu\alpha^3(\mu^2+16\mu+8) - 4\alpha(\mu^2+60\mu+20) + 96(\mu+5)] < 0. \tag{38}$$

For polynomial dis_H on argument κ we have $b_2 = -16\mu\alpha > 0,\ b_0 = 16\mu^2\alpha(\mu+4) < 0$. This means that its roots have the opposite signs, i.e. the lesser root is negative. Hence, on interval $(0,\tilde{\kappa})$[2] the expression dis_H is negative which leads to contradiction (this case study considers the non-negative discriminant of $H(p)$). Consequently, the inequality $p_{01} > -b_1/2b_2$ takes place, and the Lemma is proved.

Thus, the last two inequalities of the system (10) are equivalent to the condition $p \in (0, p_{H1})U(p_{H2}, p_{01})$.

Let us now analyze how the first of the inequalities (10) affects the found constraints on the parameter p. When considering case A1, it was established that $p_2 > p_{01}$ when $\alpha < \alpha_1$. Since for $\alpha \le 0$ the curves $p = p_{01}(\alpha)$ and $p = p_2(\alpha)$ have a single intersection point M, then, by virtue of continuity, this relation will be preserved for $\alpha \in [\alpha_1, \alpha_M)$. That is, the inequality $a_2 > 0$ will be fulfilled automatically. From the other side, it is easy to see that function $p_{01}(\alpha)$ is increasing while the function $p = p_2(\alpha)$ is decreasing. Hence, when parameter α exceed the values α_M (remind that $p_{01}(\alpha_M) = p_2(\alpha_M)$) the requirement $a_2(p,\alpha) > 0$ implies fulfillment of inequality $a_0(p,\alpha) > 0$ (the curve $p = p_2(\alpha)$ passes below the upper branch of the curve $a_0(p,\alpha) = 0$). At the same time, on the subinterval $(\alpha_M, 0)$ the system of inequalities $a_2 > 0,\ dis_H > 0$ is equivalent to single inequality $p < p_{H1}$. According to Statement 2, for this it is sufficient to make sure that $dis_H(p)|_{p=p_2(\alpha)} < 0$. The corresponding expression coincides with the expression (b8) taking with multiplier -4μ and is negative on interval (α_M, α_N). Thereby, on subinterval $(\alpha_M, 0)$ system of inequalities (10) is equivalent to inequality $p < p_{H1}$.

Remark 1. The expression in square brackets in formula (19) determines two values of parameter α : one negative and another positive. Geometrically, in plane $p\alpha$ this values correspond to abscissas of intersection points M, N of the curves $a_0(p,\alpha) = 0, a_2(p,\alpha) = 0$ (Fig. 2). Note that at this points also the equality $H(p,\alpha) = 0$ takes place.

(A3) $0 < \alpha < \alpha_F$. This is the simplest one, because inequality $a_0 > 0$ is fulfilled, and, as it was shown below, $H(\alpha,p)|_{a_2=0} < 0$. Therefore, system (10) is equivalent to single inequality $p < p_{H1}(\alpha)$.

(A4) $\alpha_N \le \alpha \le \alpha_N$. It is convenient to split this case into two subcases, depending on the position of the point N which is the common point of the curves $a_0(\alpha,p) = 0,\ a_2(\alpha,p) = 0,\ H(\alpha,p) = 0$. At the point N, the branches $p = p_{H1}(\alpha)$ and $p = p_{02}(\alpha)$, are tangent, and there are three possible cases: (a) $p_F < p_N < p_C$, (b) $p_C < p_N < p_F$, (c) $p_F = p_N = p_C$ (Fig. 3). To distinguish these cases it is sufficient to compare expressions for $p_F = 1 + \kappa/2$, and p_N which has the following form

$$p_N = \frac{1}{4}(\mu + 3\kappa - \sqrt{\mu^2 + 6\mu\kappa + \kappa^2}). \tag{39}$$

[2] Under the assumption that $\tilde{\kappa}$ is positive. If it is not, then expression (28) is definitely positive.

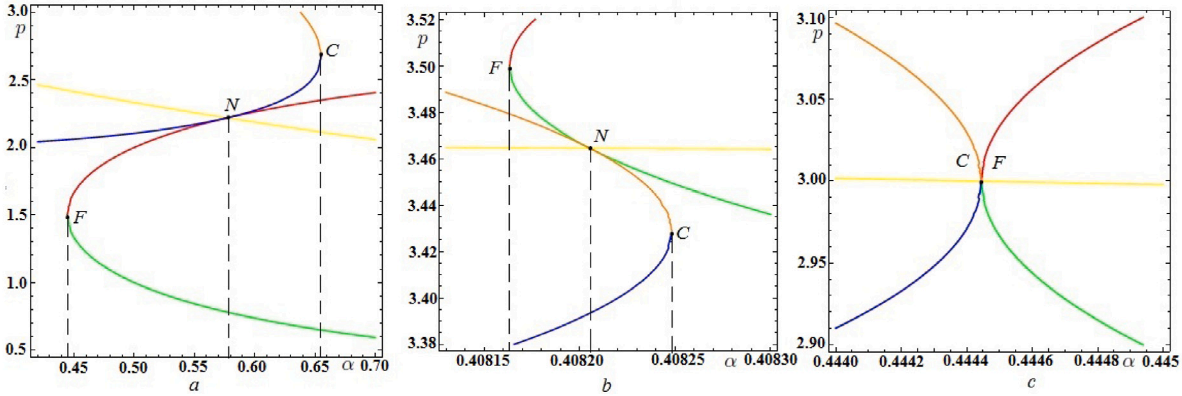

Fig. 3. Possible location of apexes F and C.: (a) $\mu = 2, \kappa = 1$; (b) $\mu = 5, \kappa = 5$; (c) $\mu = 6, \kappa = 4$.

Table 1
Stability region for parameter p.

N	α	p	Remark
$A1$	$\alpha < \alpha_A$	$p \in (0, p_{01})$	
$A2.1$	$\alpha_A \leq \alpha < \alpha_M$	$p \in (0, p_{H1}) U (p_{H2}, p_{01})$	
$A2.2$	$\alpha_M \leq \alpha \leq 0$	$p \in (0, p_{H1})$	
$A3$	$0 < \alpha < \alpha_F$	$p \in (0, p_{H1})$	
$A4$	$\alpha_F \leq \alpha < \alpha_N$	$p \in (0, p_{01}) U (p_{02}, p_{H1})$	$(2-\mu)\kappa + 2(\mu+2) > 0$
		$p \in (0, p_{01}) U (p_{02}, p_{H1})$	$(2-\mu)\kappa + 2(\mu+2) \leq 0$
$A5$	$\alpha_N \leq \alpha < \alpha_C$	$p < p_{01}$	$(2-\mu)\kappa + 2(\mu+2) \geq 0$
		$p \in (0, p_{H1}) U (p_{H2}, p_{01})$	$(2-\mu)\kappa + 2(\mu+2) < 0$
$A6$	$\alpha > \alpha_C$	$p \in (0, p_{01})$	

The difference $p_N - p_F$ is positive if and only if

$$\delta_0 = (2-\mu)\kappa + 2(\mu+2) > 0. \tag{40}$$

Remark 2. If $\delta_0 = 0$, which is possible only if $\mu > 2$, then

$$\alpha_C = \frac{1}{2\mu^2}(\mu^2 - 4), \ p_C = \frac{2\mu}{\mu - 2} = p_F. \tag{41}$$

Now we turn to the condition $a_2 > 0$. Let us show that on the interval $[\alpha_\star, \alpha_N)$ the double inequality $p_{H1} < p_2 < p_{H2}$ holds. We use Statement 2 and calculate the expression $H(\alpha, p_2)$:

$$H(\alpha, p)\,|_{p=p_2(\alpha)} = \frac{4\mu}{(2+\mu\alpha)^2}[\mu(\kappa^2 + 6\kappa + 2\mu + 8)\alpha^2 - (\mu^2 + 4\mu\kappa - \kappa^2 + 4\mu - 4\kappa)\alpha - 4\kappa]. \tag{42}$$

The expression in square brackets is a quadratic polynomial in α. Its roots are determined by the intersection points of the curves $H(\alpha, p) = 0$ and $a_2(\alpha, p) = 0$, that is, they coincide with α_M, α_N, respectively. Taking into account that the coefficient at α^2 is positive, we conclude that on the interval (α_M, α_N) the expression under consideration is negative, that is, the solution to the system of inequalities $H > 0, a_2 > 0$ is the set $p < p_{H1}$.

Depending on which sign the expression $\delta_0 = (2-\mu)\kappa + 2(\mu+2)$ has, the solution of the system (10) for $\alpha \leq \alpha_N$ is defined as follows: $p \in (0, p_{01}(\alpha)) \bigcup (p_{02}(\alpha), p_{H1}(\alpha))$ if $\delta_0 > 0$ or $p < p_{H1}$ if $\delta_0 < 0$. Similarly, one can verify that

$$a_0(\alpha, p)\,|_{p=p_2(\alpha)} = -\frac{[\star]}{(2+\mu\alpha)^2}, \tag{43}$$

where symbol $[\star]$ denotes the expression in square brackets on the right side of formula (42). Taking into account the above argumentation, we conclude that for $\alpha > \alpha_N$, the double inequality $p_{01} < p_2 < p_{02}$ holds.

Then the stability conditions (10) in the case $\delta_0 > 0$ are equivalent to the inequality $p < p_{01}$, and in the case $\delta_0 < 0$– the unity of inequalities $p < p_{H1}$ and $p_{H2} < p < p_{01}$.

(A5) $\alpha > \alpha_C$. In this case, the third inequality (10) is satisfied automatically; moreover, the double inequality $p_{01} < p_2 < p_{02}$ takes place. Therefore, the system of inequalities (10) is equivalent to the single inequality $p < p_{01}$.

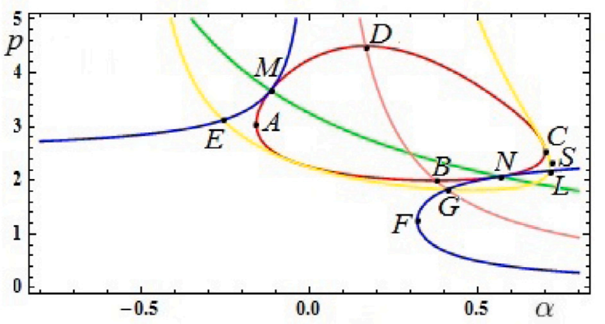

Fig. 4. The domain of destabilization caused by small damping (zone between yellow and red lines). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

4. Analysis of stability conditions for system $(6)_{lin}$

In this section we consider the weakly damping case ($0 < \varepsilon \ll 1$) and how the stability conditions change comparing with results presented in Table 1.

The necessary and sufficient conditions of stability according to Lienard–Chipart criterion [38] are

$$a_3 > 0, a_1 > 0, a_0 > 0, \Delta_3 = \begin{vmatrix} a_3 & a_1 & 0 \\ a_4 & a_2 & a_0 \\ 0 & a_3 & a_1 \end{vmatrix} > 0. \tag{44}$$

After substitution expression for coefficients a_j $(j = 1, \dots, 4)$ we have $\Delta_3 = \Delta + O(\varepsilon^2)$ where

$$\Delta = h_2 p^2 - 2h_1 p + h_0,\ \ h_2 = \alpha[\mu(\mu+2)(d+2)\alpha + (d-\mu)(\mu+d+4)],$$

$$h_1 = \alpha\{d^2 + [\mu^2 + 3\mu + 4 - \kappa(\mu+1)]d - \kappa(3\mu+4)\} + d^2 + (\mu+\kappa+4)d + 2\kappa(\mu+4),$$

$$h_0 = 4d^2 + [(\mu-\kappa)^2 + 8(\mu+2)]d + 4\kappa^2. \tag{45}$$

In the same spirit as it was done in Section 3 consider in the plane (α, p) three-parameter families of curves $a_0(\alpha, p) = 0,\ p = p_1(\alpha),\ \Delta(\alpha, p) = 0$.

Substituting $p = (d+\kappa)/\alpha(d+2)$ into expression for a_0, we obtain

$$\frac{1}{(d+2)^2}\Big[\frac{(d+k)^2}{\alpha} - 2d^2 - d(\kappa^2+4) - 2\kappa^2\Big],\ \ \alpha \neq 0. \tag{46}$$

Therefore, the curves $p = p_1(\alpha)$ and $a_0(\alpha, p) = 0$ have a single intersection point G with coordinates

$$\alpha_G = \frac{(d+\kappa)^2}{(d+2)(\kappa^2+2d)} > 0,\ \ p_G = \frac{\kappa^2+2d}{\kappa+d}. \tag{47}$$

Since the equalities $a_0 = 0,\ a_1 = 0$ entail the equality $\Delta = 0$, then the point G belongs to the curve $\Delta(\alpha, p) = 0$.

Point G belongs to the branch $p = p_{01}(\alpha)$ if $p_G < 1 + \kappa/2$ and the branch $p = p_{02}(\alpha)$ otherwise. Writing down the difference $p_G - 1 - \kappa/2$, we get

$$\frac{(\kappa-2)(\kappa-d)}{2(\kappa+d)}. \tag{48}$$

Thus, the curve $p = p_1(\alpha)$ intersects the upper branch of the curve $a_0(\alpha, p) = 0$, if

$$\kappa < \min(2, d)\ \ or\ \ \kappa > \max(2, d), \tag{49}$$

and the lower branch of this curve if

$$\min(2, d) < \kappa < \max(2, d). \tag{50}$$

The curves $a_0(\alpha, p) = 0$ and $\Delta(\alpha, p) = 0$ have, in addition to the point G, two more intersection points E and L (Fig. 4).

Their abscissas can be found by equating to zero the resultant of the corresponding polynomials in the argument p

$$rez_1(\alpha) = \alpha[(d+2)(\kappa^2+2d)\alpha - (d+\kappa)^2](g_2\alpha^2 + g_1\alpha + g_0), \tag{51}$$

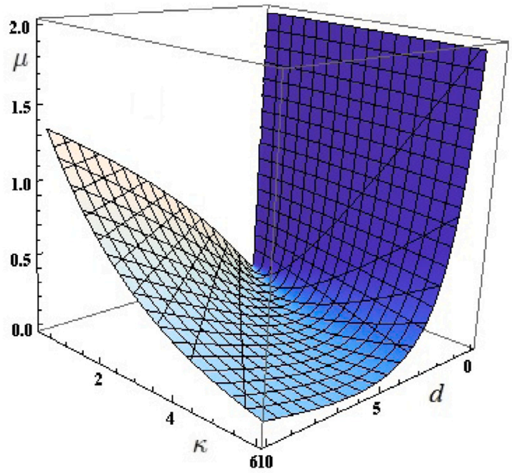

Fig. 5. The surface obtained based on (56).

where

$$g_2 = \mu(\mu+2)[(\kappa+2)(\kappa+4)d + 2(\mu+\kappa+4)(\mu+2\kappa+4)],\ g_1 = \kappa(\kappa+4)d^2 + 2[(\mu+4)\kappa^2 + \\ + 2(-\mu^2+\mu+8)\kappa - 2\mu(\mu+4)]d + (4\kappa-\mu^2-4\mu)[2(\mu+2)\kappa+(\mu+4)^2], \tag{52}$$

$$g_0 = -4\kappa(d+\mu+4)^2.$$

Since $g_2 > 0, g_0 < 0$, then the polynomial $g_2\alpha^2 + g_1\alpha + g_0$ has two real roots of opposite signs. We denote these roots by $\alpha_E (< 0), \alpha_L$. The corresponding expressions are

$$\alpha_E = -\frac{1}{2g_2}(g_1 + \sqrt{g_1^2 + 4g_2g_0}) < 0,\ \alpha_L = \frac{1}{2g_2}(-g_1 + \sqrt{g_1^2 + 4g_2g_0}) > 0. \tag{53}$$

The ordinates of the points E, L are the roots of the polynomial

$$\psi_2(p) = 2(\mu+d+4)p^2 - [(3\kappa+8)d + 2\kappa(\mu+6) + (\mu+4)(\mu+8)]p + \\ + (\kappa+2)(\kappa+4)d + 4\kappa^2 + 6\kappa(\mu+4) + 2(\mu+4)^2\ (p_E > p_G > 0). \tag{54}$$

Let us now turn to the equation $\Delta(\alpha, p) = 0$. Note that the curves of this family degenerate if the expression

$$dis_\Delta(\alpha) = h_1^2 - h_2h_0 = 4(\mu+d+4)^2\{[(d-\kappa)^2 - 2d\mu(\mu+\kappa+2)]\alpha^2 - \\ [2d^2 - (\mu^2+4\mu-\kappa^2)d + 2\kappa^2]\alpha + (d+\kappa)^2\} \tag{55}$$

is negative.

The discriminant of the quadratic polynomial $dis_\Delta(\alpha)$ is

$$16d(\mu+\kappa)^2\{4(d^2+\kappa^2) + d[(\mu-\kappa)^2 + 8\mu + 16]\} > 0, \tag{56}$$

hence $dis_\Delta(\alpha)$ has two different real roots α_3, α_4 $(\alpha_3 < \alpha_4)$. Note that over a wide range of parameter values the expression $(d-\kappa)^2 - 2\mu d(\mu+\kappa+2)$ is negative (Fig. 5), therefore $\alpha_3\alpha_4 < 0$.

Consider the question of the sign of the expression $h_2(\alpha, \mu, \kappa, d)$. With respect to the parameter α it is negative if

$$\min(0, \alpha_5) < \alpha < \max(0, \alpha_5),\ \alpha_5 = \frac{(\mu-d)(\mu+d+4)}{\mu(\mu+2)(d+2)} \tag{57}$$

and is positive outside the specified interval. Insofar as

$$dis_\Delta|_{\alpha=\alpha_5} = \{\frac{\kappa[(\mu+1)d^2 + 4(\mu+1)d - \mu^2] + d[\mu(\mu^2+5\mu+8) - d^2 - 4d]}{\mu(\mu+2)(d+2)}\}^2 \geq 0, \tag{58}$$

then, according to Statement 2, $\alpha_3 \leq \alpha_5 \leq \alpha_4$, and the domain $h_2 \leq 0$ in the parameter space α, μ, κ, d belongs to the domain $dis_\Delta \geq 0$. As a consequence, if $dis_\Delta < 0$, then $h_2 > 0$, and $\Delta > 0$.

The roots of the polynomial $\Delta(\alpha, p)$ with respect to parameter p are determined by formula

$$p_{\Delta j} = \frac{1}{h_2}(h_1 + (-1)^j\sqrt{h_1^2 - h_2h_0}),\ j = 1, 2. \tag{59}$$

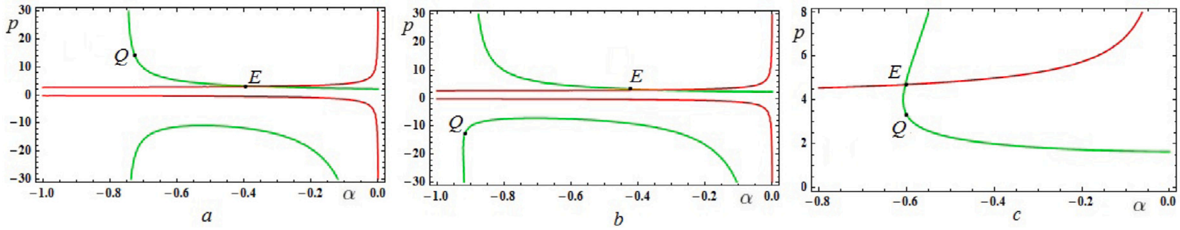

Fig. 6. Different variants for intersection the curves $\Delta(\alpha,p)=0$ and $a_1(\alpha,p)=0$: (a) $\mu=2,\ \kappa=0.6,\ q=6$; (b) $\mu=2,\ \kappa=0.5,\ q=7$; (c) $\mu=2,\ \kappa=2,\ q=3$.

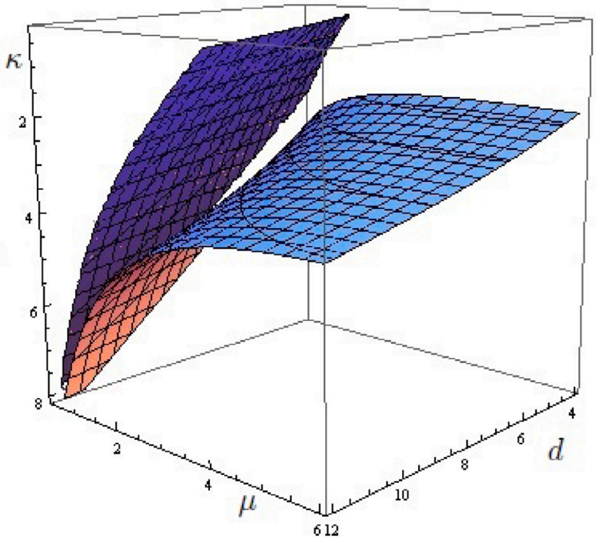

Fig. 7. Surface in parameter space where the equality $p_E=p_Q$ takes place.

Remark 3. In fact, for the purpose of analysis the stability conditions (44) we can exclude the cases $\alpha_5=\alpha_j$ $(j=3,4)$ from the consideration. Indeed, if such equality takes place, then $h_2=0,\ h_1^2-h_0h_2=0$, therefore $h_1=0$. Taking into account that $h_0>0$, one can see that the requirement $\Delta>0$ is fulfilled.

Let us denote as Q and S points where two branches $p=p_{\Delta_1}(\alpha),\ p=p_{\Delta_2}(\alpha)$ meet each other. Their abscissas, as it follows from the formula (55) are α_3,α_4, and the corresponding values for ordinates are determined as following expressions

$$p_Q=\left(\frac{h_1}{h_2}\right)\Big|_{\alpha=\alpha_3},\ \ p_S=\left(\frac{h_1}{h_2}\right)\Big|_{\alpha=\alpha_4}. \tag{60}$$

Let $\alpha\le 0$. Then the inequality $a_1(\alpha,p)>0$ is fulfilled. On interval $\alpha\in(-\infty,\alpha_E)$ two kinds of solution for system $a_0>0,\Delta>0$ are possible. If for given values of parameters μ,κ,d the value of the expression p_E does not exceed the value of p_Q (the upper branch of the curve $a_0(\alpha,p)$ intersects with the lower branch of the curve $\Delta(\alpha,p)=0$), then stability conditions are fulfilled if and only if $p<p_{01}(\alpha)$ while $\alpha<\alpha_E$. If $\alpha\in[\alpha_E,0]$, then these conditions are equivalent to inequality $p<p_{\Delta 1}(\alpha)$ (Fig. 6a). The same result takes place when $p_Q\le 0$ (Fig. 6b). The explicit expression for p_Q is bulky enough and is formed by substitution the lesser root of the polynomial (55) (α_3, which is the irrational expression) into formula (60), with considering formulas (45). The surface $p_E=p_Q$ is presented in Fig. 7.

If parameters μ,κ,d satisfy the condition $0<p_Q<p_E$ (Fig. 6c), then stability conditions may be presented in the following form

$$\begin{gathered} p<p_{01}(\alpha)\ if\ \alpha\le\alpha_Q,\\ p\in(0,p_{\Delta 1})U(p_{\Delta_2},p_{01})\ if\ \alpha\in[\alpha_Q,\alpha_E),\\ p<p_{\Delta 1}(\alpha)\ if\ \alpha\in[\alpha_E,0]. \end{gathered} \tag{61}$$

Suppose that $\alpha\in(0,\alpha_G)$. Consider the system of inequalities

$$p>p_1(\alpha,\kappa,d),\ \ \Delta(\alpha,p,\mu,\kappa,d)>0. \tag{62}$$

Let us calculate the expression

$$\Delta|_{p=p_1}=\left(1+\frac{\mu+2}{d+2}\right)^2\left[(d+2)(\kappa^2+2d)-\frac{(d+\kappa)^2}{\alpha}\right]. \tag{63}$$

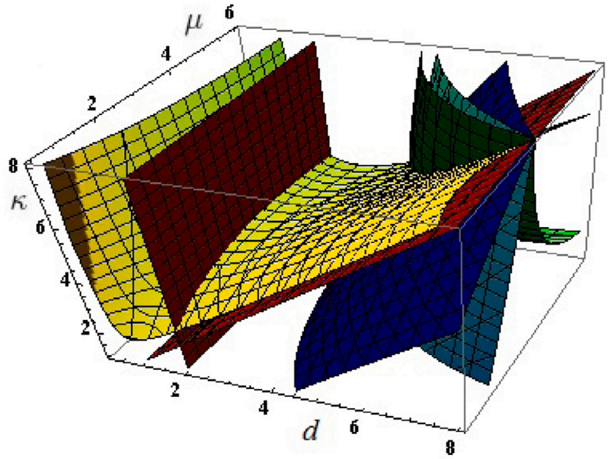

Fig. 8. Partition of the parametric space $\mu\kappa d$ defining the setting of G, L, F, S ordinates hierarchies.

It is negative because the expression in square brackets is negative in the range in question. Then, if $h_2 > 0$, then this means (statement 2) that the value $p = p_1$ is located between the roots of the polynomial $\Delta(p)$, and the system (62) is equivalent to the inequality $p < p_{\Delta 1}$.

If $h_2 < 0$, then p_1 is greater than the larger of the roots of the polynomial $\Delta(p)$, which is again determined by expression $p_{\Delta 1}$ according to formula (59), since the factor $1/h_2$ has changed its sign. Therefore, in this case, system (62) is equivalent to the inequality $p < p_{\Delta 1}$.

The explicit expression for p_F is rather cumbersome and is not presented here.

Remark 4. Recall that the boundary of the stability region was obtained under the assumption of weak damping with an error $O(\varepsilon^2)$.

For the convenience of the subsequent explanation, we introduce the following label: W will denote the one of the points G, L whose ordinate is the lesser. The surface $p_G = p_L$ separates the first octant of parametric space μ, κ, d on two parts, thus for different values of these parameters W may represent either point G or L.

Note that the solution of the system of inequalities (44) with respect to the parameter p essentially depends on the values of the parameters μ, κ, d. Typologically, the restrictions on parameter p, which satisfy the stability conditions, can be divided into three cases:

(C1) point W belongs to branches $p = p_{\Delta 1}(\alpha)$ and $p = p_{01}(\alpha)$;
(C2) point W belongs to branches $p = p_{\Delta 1}(\alpha)$ and $p = p_{02}(\alpha)$;
(C3) point W belongs to branches $p = p_{\Delta 2}(\alpha)$ and $p = p_{01}(\alpha)$.

Remark 5. To distinguish these cases we need to compare values of $p_F = 1 + \kappa/2$ and p_G, p_L, p_S which are determined according to formulas (47), (54) and (60) respectively. The comparison of each pair may be presented as a 3D-surface in the parameter space μ, κ, d (Fig. 8). These surfaces separates the first octant onto domains each of which corresponds to some hierarchy of values p_F, p_G, p_L, p_S. In fact, there are 14 possible combinations (Fig. 9d).

Consider these three cases in sequence.

(C1) From the viewpoint of the analysis of stability conditions, the simplest case is $p_G < p_L$. Since on the interval $(0, \alpha_G)$ the condition $\Delta|_{p=p_1} < 0$ holds, then the system $\Delta > 0, a_1 > 0$ is equivalent to the inequality $p < p_{\Delta 1}$. In this case, the condition $a_0 > 0$ is satisfied automatically. Similarly, on the interval $\alpha \geq \alpha_G$, we have

$$a_0(\alpha, p)\,|_{p=p_1(\alpha)} = \frac{(\mu + d + 4)^2}{\alpha(d+2)^2}[-(d+2)(\kappa^2 + 2d)\alpha + (\kappa + d)^2] = = \frac{(d+2)(\kappa^2 + 2d)(\mu + d + 4)^2}{\alpha(d+2)^2}(\alpha_G - \alpha) \geq 0, \tag{64}$$

and the system of inequalities $a_0 > 0, a_1 > 0$ is equivalent to the condition $p < p_{01}(\alpha)$, in this case, the inequality $\Delta > 0$ takes place.

Let us now turn to the case $p_L < p_G$. In the case of the $LFGS$ hierarchy, the solution of subsystem of inequalities $a_1 > 0,\ \Delta > 0$ has the form $p < p_{\Delta_1}(\alpha)$ (since the point G belongs to the lower branch of the curve $\Delta(\alpha, p) = 0$). In this case, taking into account the inequality $a_0 > 0$ we have the restriction $p < p_{01}(\alpha)$. Thus, stability conditions are fulfilled if and only if $p < p_{\Delta_1}(\alpha)$ while $\alpha \in (0, \alpha_L)$ and $p < p_{01}(\alpha)$ while $\alpha \geq \alpha_L$. If the hierarchies $LFSG$ or $LSFG$ take place, point G belongs to the upper branches of the curves $\Delta(\alpha, p) = 0,\ a_0(\alpha, p) = 0$. However, the fulfillment of the inequalities $p > p_{\Delta 2}, p < p_1$ leads to the inequality $a_0 < 0$, that is, the violation

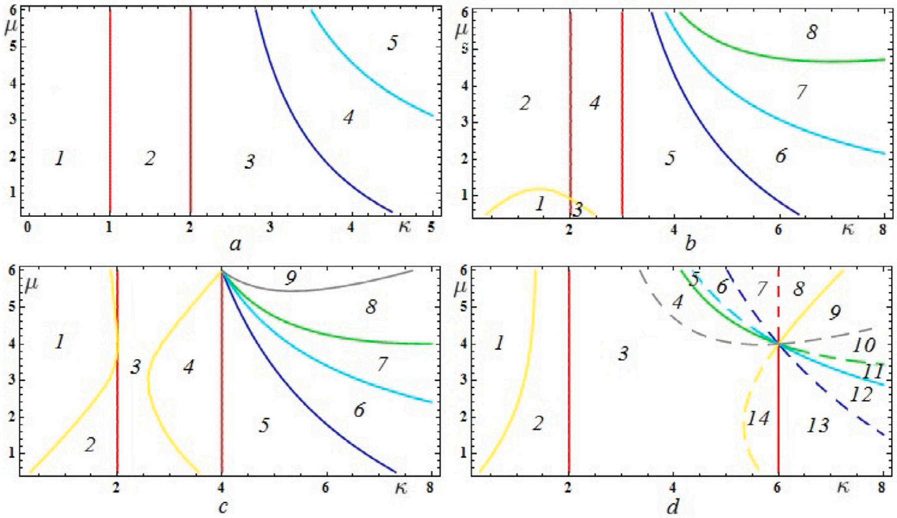

Fig. 9. Surfaces sections by planes $d = const$ from Fig. 8: (a) $d = 1$; (b) $d = 3$; (c) $d = 4$; (d) $d = 6$. This cut contains the most of different hierarchies for values p_F, p_G, p_L, p_S. In numerical order they are: $FGLS, FGSL, GFSL, GSFL, SGLF, SLGF, SLFG, LSFG, LFSG, LFGS, FLGS, FGLS$ (the same as in domain 1), and $GFLS$. Crossing the dash line does not change the case (C1, C2 or C3).

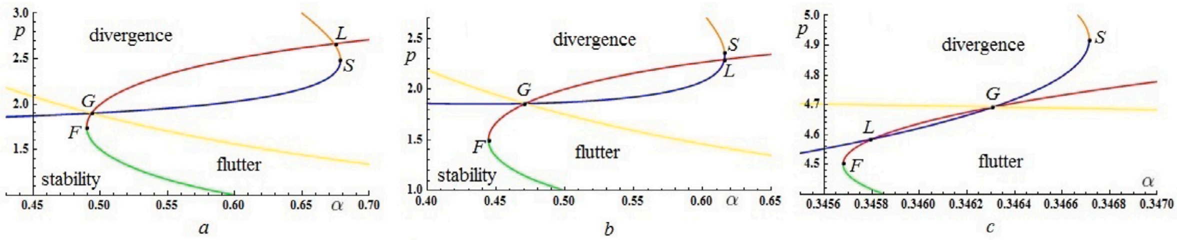

Fig. 10. Case C2. The possible combinations of mutual location the "governing points" G and L for $d = 6$: (a) domain 2 ($\mu = 1, \kappa = 1.5$); (b) domain 1 ($\mu = 4, \kappa = 1$); (c) domain 12 ($\mu = 3, \kappa = 7$).

of stability conditions occurs. Similarly, the fulfillment of inequalities $p > p_{02}, p < p_1$ implies $\Delta < 0$. Thus, the stability conditions are equivalent to the inequalities

$$\begin{gathered} p < p_{\Delta 1}(\alpha) \ if \ \alpha \in (0, \alpha_W], \\ p < p_{01}(\alpha) \ if \ \alpha > \alpha_W. \end{gathered} \tag{65}$$

(C2) If $p_G < p_L$, then the reasoning given when considering case C1 is valid with the difference that on the interval $[\alpha_F, \alpha_G]$ the fulfillment of the inequality $p < p_{\Delta 1}$ does not guarantee the fulfillment of the condition $a_0 > 0$. In other words, the stability conditions have the form

$$\begin{gathered} p < p_{\Delta 1}(\alpha) \ if \ \alpha \in (0, \alpha_F), \\ p \in (0, p_{01}(\alpha)) U (p_{02}(\alpha), p_{\Delta 1}(\alpha)) \ if \ \alpha \in [\alpha_F, \alpha_W), \\ p < p_{01}(\alpha) \ if \ \alpha \geq \alpha_W. \end{gathered} \tag{66}$$

The same conditions hold for $p_L < p_G$, however now the point L is acting as W (Fig. 10).

(C3) Carrying out arguments similar to those given above, we obtain the conditions for stability

$$\begin{gathered} p < p_{\Delta 1}(\alpha) \ if \ \alpha \in (0, \alpha_W], \\ p \in (0, p_{\Delta 1}(\alpha)) U (p_{\Delta 2}(\alpha), p_{01}(\alpha)) \ if \ \alpha \in [\alpha_W, \alpha_S), \\ p < p_{01}(\alpha) \ if \ \alpha \geq \alpha_S. \end{gathered} \tag{67}$$

The typical cases are shown in Fig. 11.

As it follows from the results of Section 3, the admissible value of the parameter p, corresponding to the critical load P_{crit}, depends significantly on the angle between the axis of the second link of the pendulum and the direction of action of the external load. This value also depends on the ratios of masses, stiffnesses and damping coefficients. In this section, we will highlight some of the salient features. We will assume that the parameters α and μ are determined and discuss the influence of the parameters d and κ on the value p_{crit}.

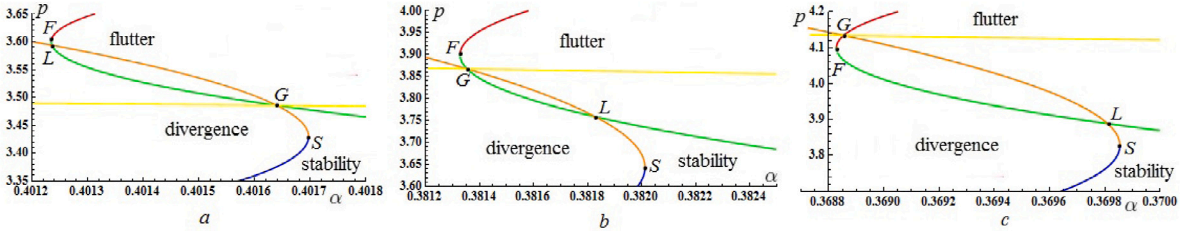

Fig. 11. Case C3 for $d = 6$: (a) domain 6 ($\mu = 4.8, \kappa = 5.2$); (b) domain 7 ($\mu = 5, \kappa = 5.8$); (c) domain 8 ($\mu = 5, \kappa = 6.2$).

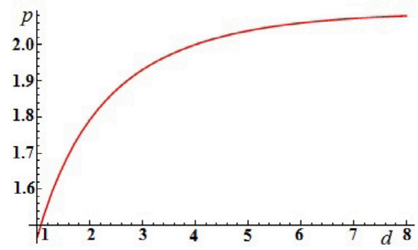

Fig. 12. Dependence the value of p_{crit} on damping ratio (Ziegler's case).

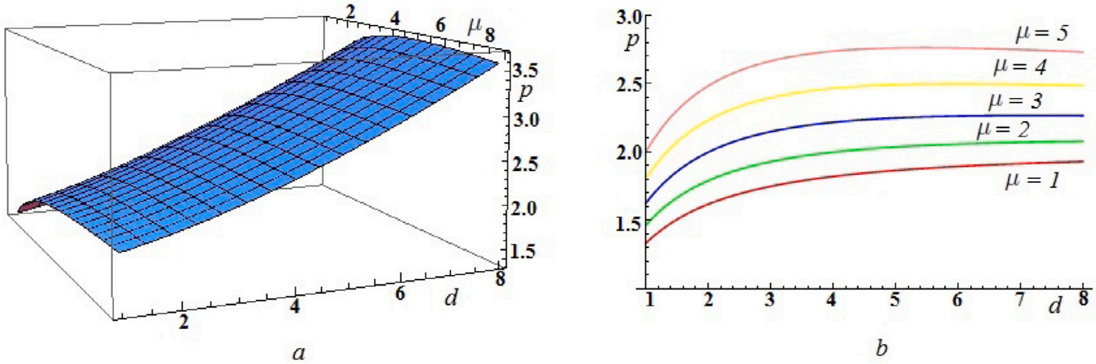

Fig. 13. (a) The surface $p = p^0_{crit}(d, \mu)\mid_{\kappa=1}$ in parameter space $\mu d p$; (b) dependence the critical load values from damping ratio for different values of μ.

(1) As for the influence of the damping ratio d, then, as noted in the work of Herrmann [8] and a number of subsequent publications [12,32,34,37], for the case $\alpha = 0,\ \mu = 2,\ \kappa = 1$ the value of p_{crit} is determined by the expression

$$\widetilde{p}^0_{crit} = \frac{4d^2 + 33d + 4}{2(d^2 + 7d + 6)}. \tag{68}$$

The latter takes on values greater than 2 starting from about $d = 4$ (Fig. 12) and reaches a maximum of ≈ 2.086[3] at $q = 4 + 5\sqrt{2}$, which cancels the destabilization effect.

This tendency for d is valid in our study. For limit case $\alpha = 0$ the expression for p_{crit} has the following form

$$p^0_{crit} = \frac{(d+4)\kappa^2 - 2\mu d\kappa + d(\mu+4)^2 + 4d^2}{2(\mu + d + 4)(\kappa + d)}, \tag{69}$$

and typical view of the surface $p = p^0_{crit}(d, \mu)\mid_{\kappa=1}$ is presented in Fig. 13.

This monotonic (limited) increasing of p_{crit} with increased value of d continues for nonzero values of $\alpha < \alpha_\star$ (Fig. 14).

The numerical testing based on integrating the nonlinear equations (4) confirms this conclusion. In Fig. 15 there are presented time histories for φ_1, φ_2 and projections of phase portrait for variable φ_2 for the following values of parameters: $\mu = 2, \kappa = 1, \alpha = 0.3$ and different damping coefficients. For the cases (a) and (b) these coefficient are respectively $d_1 = 0.12, d_2 = 0.04$ and $d_1 = 0.2, d_2 =$

[3] For comparison, with $d = 1$ we have $p_{crit} \approx 1.464$.

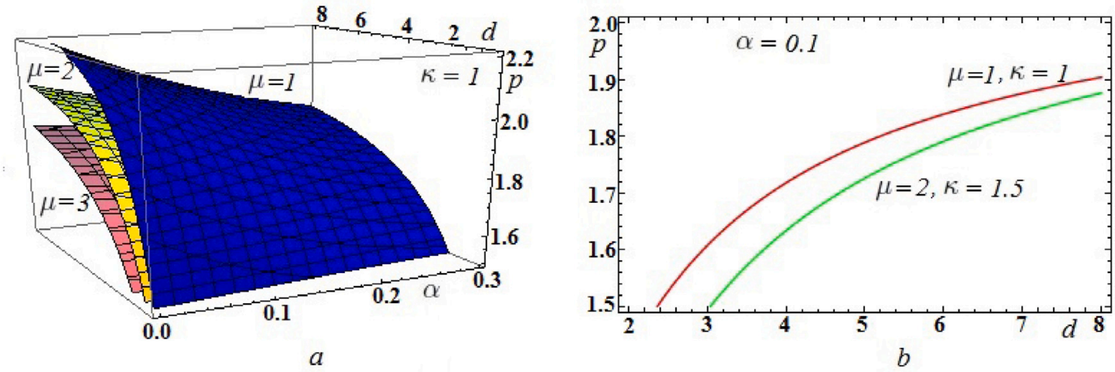

Fig. 14. Dependence the value of p_{crit} on damping ratio (imperfect follower force): (a) surfaces $p = p_{crit}(d, \alpha)$ for fixed stiffness ratio $\kappa = 1$ and different mass ratios; (b) monotonic increase p_{crit} on damping ratio.

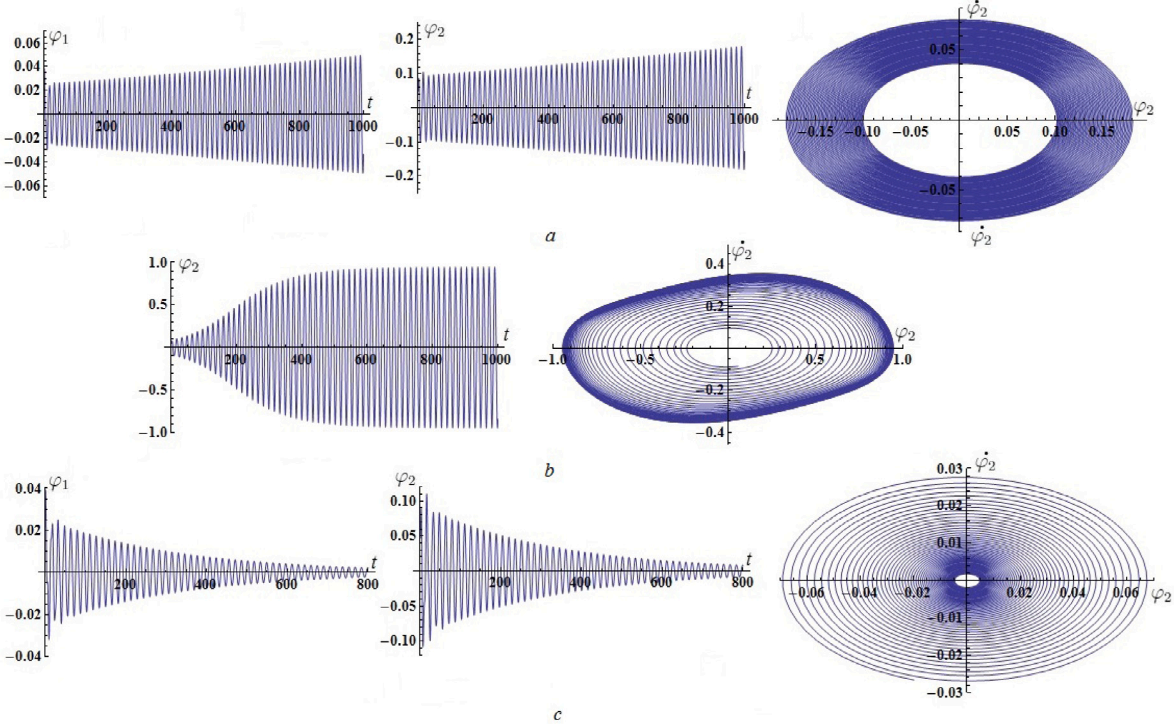

Fig. 15. Change in behavior by varying the damping ratio for $\mu = 2, \kappa = 1, \alpha = 0.3$. Initial values are $\varphi_1(0) = 0.025, \varphi_2(0) = 0.07,\ \ \dot{\varphi}_1(0) = 0.02, \dot{\varphi}_2(0) = -0.03$: (a) $d_1 = 0.12, d_2 = 0.04$; (b) $d_1 = 0.2, d_2 = 0.1$; (c) $d_1 = 0.1, d_2 = 0.02$. On (a) slow rate of increase (flutter), on (b) strong rate of increase and asymptotical stability on (c) are present.

0.1, and the origin is unstable. Although for case (b) the damping is higher, but the damping ratio is smaller $d = 2 < 3$), and the increase of amplitude is stronger. Contrary, in case (c) the damping coefficients are smaller ($d_1 = 0.1, d_2 = 0.02$), but origin is asymptotically stable, because the damping ratio is higher ($5 > 3 > 2$).

(2) Also, it seems interesting to study the influence of the parameter κ on the value of the critical load. Here we restrict ourselves to discussing the situation near the boundary of the flutter instability zone ($0 \leq \alpha < \alpha_\star$). As it was shown in Section 4, this boundary is determined by the lower branch of the curve $\Delta_3(\alpha, p) = 0$. Considering that the value of $\alpha_\star$ does not exceed 4/9 if $\kappa \leq 1$, then we can use the asymptotic representation for $p_{\Delta_1}(\alpha)$:

$$p = p^0_{crit} + \alpha \frac{p^0_{crit}}{4(\kappa + d)} \{[\mu(3d + 8) + (d + 4)^2]\kappa^2 + 2\mu d(d - \mu)\kappa - \\ - \mu d[d(3\mu + 8) + (\mu + 4)^2]\} + O(\alpha^2). \tag{70}$$

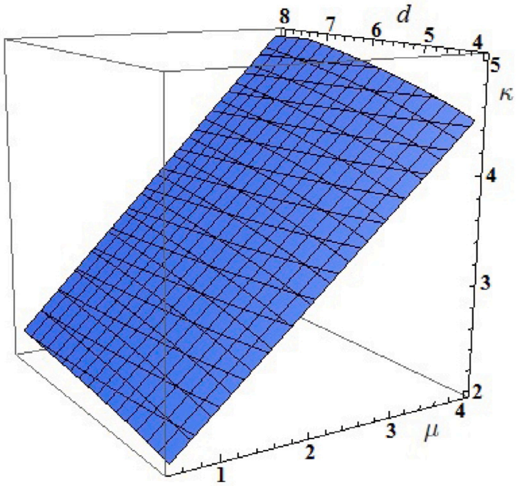

Fig. 16. The surface $\kappa = \kappa_2(\mu, d)$.

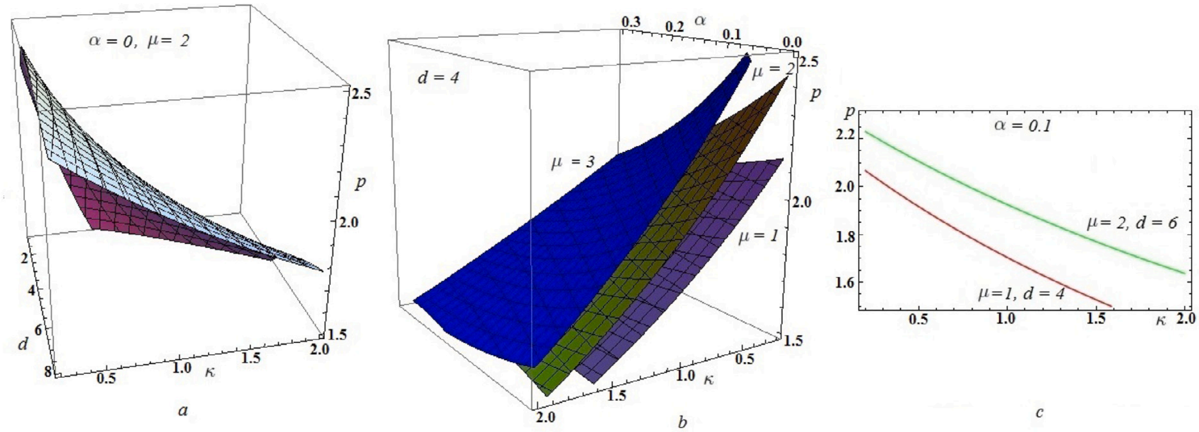

Fig. 17. Dependence the value of p_{crit} on stiffness ratio for constant damping ratio: (a) the view of surface $p = p^0_{crit}(d, \kappa)$ for perfect follower force; (b) dependence p_{crit} on stiffness ratio with fixed damping ratio and different values of μ and non-zero α; (c) the benefit of decreasing the stiffness ratio (with fixed mass and damping ratios) for increase of p_{crit}.

As noted above, values for damping ratio d that are several times greater than unity are effective. In particular, for $d \geq 4,\ \kappa \leq 2$ expression (69) is a decreasing function of the argument κ. Indeed, we have for derivative the following expression

$$\frac{dp^0_{crit}}{d\kappa} = \frac{(4+d)\kappa^2 + 2d(4+d)\kappa - d[\mu^2 + 2(d+4)\mu + 4(d+4)]}{2(d+\kappa)^2(\mu+d+4)}. \tag{71}$$

The denominator of the fraction (71) is positive, and the numerator is negative if

$$\kappa < \kappa_2 = \sqrt{d}\left(\frac{\mu}{\sqrt{d+4}} + \sqrt{d+4} - \sqrt{d}\right). \tag{72}$$

As one can see from Fig. 16, the expression κ_2 takes on values greater than two for $\mu \geq 0.5$. The other terms of the asymptotic expansion (70) does not change this dependence p_{crit} from κ (Fig. 17).

The growth of critical value of external force achieved by softening the spring at the base of the pendulum is not very large, however, depending on value of the parameter α, the gain of 10–15 percent is possible. In Fig. 18 the results of numerical integration of motion equations for variable φ_2 are presented. For parameters $\mu = 2, \alpha = 0.2, d_1 = 0.12, d_2 = 0.02$ different values of κ were tested. For $\kappa = 1$ the value for $p_{crit} = p_{\Delta_1}$ is 1.904 (Fig. 18a). As the value of κ is decreasing, the value of p_{crit} is increasing: $p_{crit} = 2.015$ with $\kappa = 0.5$ and p_{crit} is equal to 2.104 when $\kappa = 0.2$.

(3) It should also be noted that, in contrast to the case of a perfect follower force ($\alpha = 0$), there are intervals of variation of the parameter α, in which the following phenomenon takes place: when $p = p_{\Delta_1}(\alpha) - \varepsilon_1$ ($\varepsilon_1 > 0$ is infinitesimally small) the equilibrium

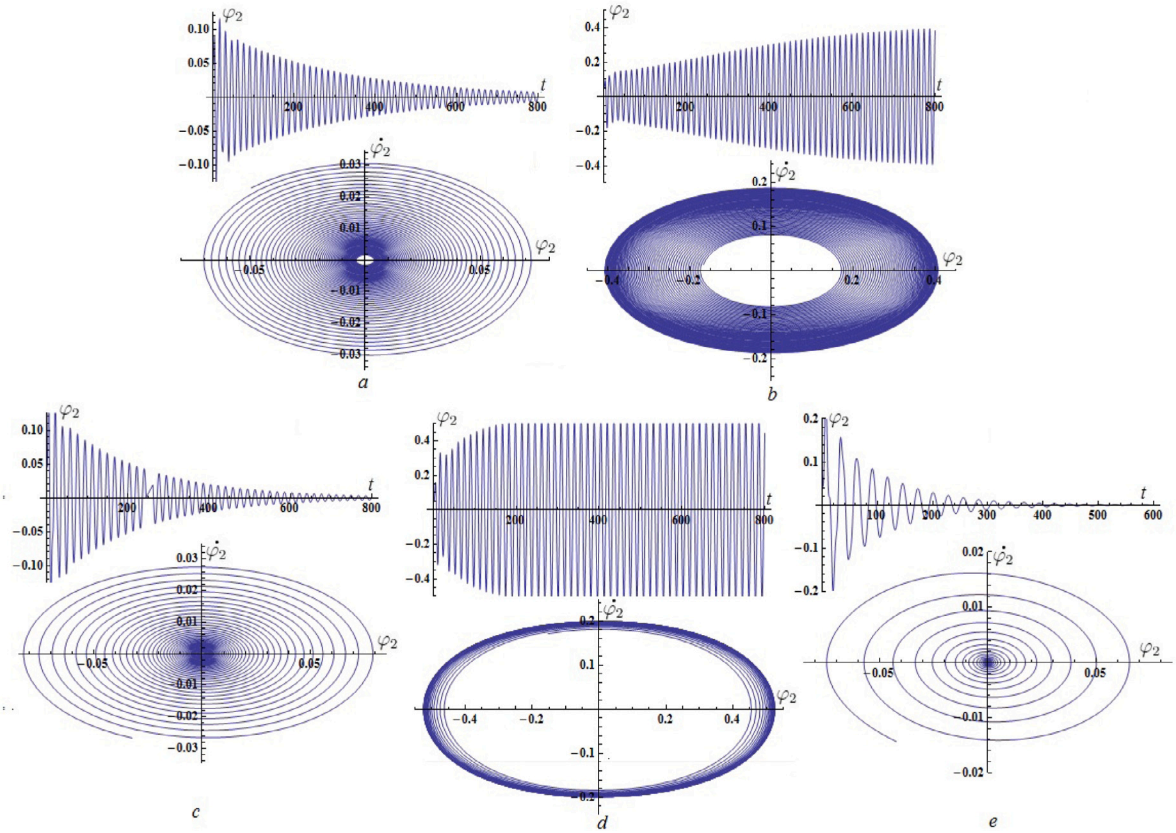

Fig. 18. Numerical validation of increasing the value p_{crit} with decreasing the stiffness ratio for $\mu = 2, \alpha = 0.3, d_1 = 0.12, d_2 = 0.02$: (a) $\kappa = 1, p = 1.8$; (b) $\kappa = 1, p = 1.9$; (c) $\kappa = 0.5, p = 1.9$; (d) $\kappa = 0.5, p = 2$; (e) $\kappa = 0.2, p = 1.95$.

position under study is asymptotically stable, but when the load decreases, stability is lost when the value $p_{02}(\alpha)$ is achieved and equilibrium regain stability under $p_{01}(\alpha)$. This loss of stability as a rule is soft (the stable periodic orbits appear).

To illustrate this feature, we will consider a special case $\mu = 2, \kappa = 1$. Then we have: $\alpha_\star \approx 0.444$, $\alpha_G \approx 0.463$, and for our test we take $\alpha = 0.45$. The corresponding values for p_{Δ_1}, p_{02} and p_{01} are respectively $1.795, 1.665, 1.335$. The results of numerical integration are presented in Fig. 19.

For the value $p = 1.8$ very weak flutter takes place, the eigenvalues are $0.0001 \pm 0.1265i, -0.050 \pm 0.934$. For the value $p = 1.78$ the equilibrium becomes asymptotically stable (Fig. 19b), the first pair of eigenvalues now have negative real part -0.0004 (the second pair chances slightly). For the value $p = 1.67$ the eigenvalues are $-0.0023 \pm 0.0151i$, $-0.0477 \pm 1.037i-$ equilibrium is asymptotically stable. But at $p = 1.66$ stability is lost, we have two real eigenvalues of opposite signs (divergent instability). The trajectory run away from zero to other point of attraction. This behavior takes place up to value $p = 1.335$, the limit value for φ_2 is gradually decreasing (Fig. 19e). Eventually, the origin returns to stable state (Fig. 19f), the eigenvalues for $p = 1.33$ are the following: $-0.0045 \pm 0.012i$, $-0.0455 \pm 1.253i$.

For rigorous analysis of such behavior it is not sufficient dealing with the linearized system $(6)_{lin}$, because the nonlinear terms $\mathbf{F}(\boldsymbol{\varphi})$ may drastically change the result. Such case requires the separate study and is the subject of the future work.

5. Conclusion

In the paper the stability problem for double pendulum subjected to imperfect follower force is studied. Unlike other works on this subject, all three ratios: the mass, damping and stiffness presumed unknown. Their influence on the value of critical load is investigated. Stability conditions for cases of undamped and weakly-damped systems are analyzed. The geometrical interpretation of these conditions is given.

In particular, it seems interesting for applications the possibility of increasing the value of critical load by decreasing the stiffness ratio, i.e. softening the hinge at the base of the pendulum. Also the phenomenon of stability transition with changing the value of load is discovered, when the stable equilibrium under some value $p = p_1$ may become unstable for values $p < p_1$. This phenomenon in some sense is similar to cases discussed in papers [3,39]. Its understanding require the nonlinear analysis of the equations and is the subject of further research.

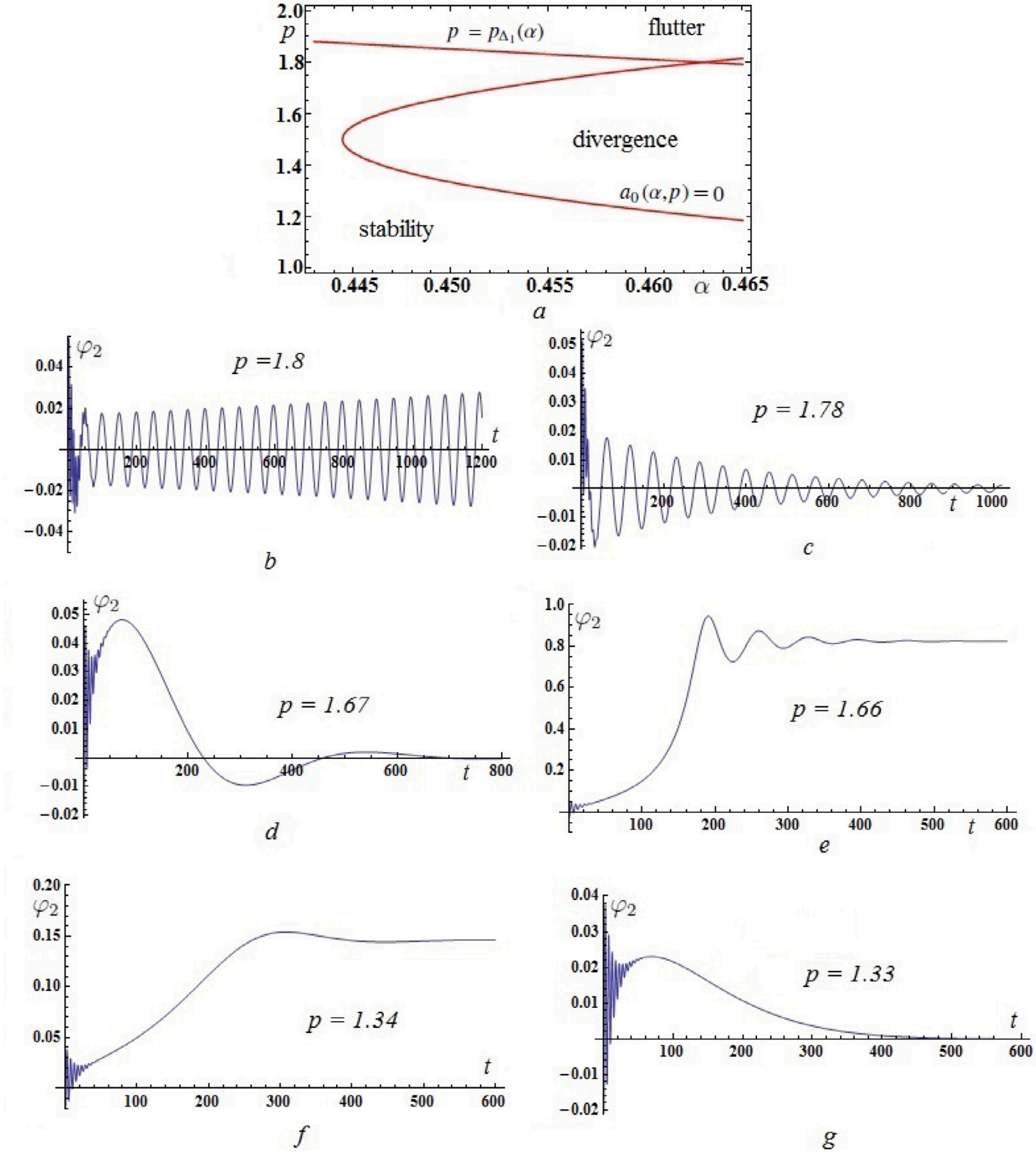

Fig. 19. Stability transition “stable - unstable - stable” with decreasing the value of p : (a) Determining the values of p where such transition occurs; (b) weak instability (flutter) at $p = 1.8$; c, (d) stability at $p = 1.78$ and $p = 1.67$ (near the boundaries of upper stability zone); d, (e) instability (soft divergence) at $p = 1.66, p = 1.335$ respectively; (f) the lower stability zone, $p = 1.33$.

CRediT authorship contribution statement

Volodymyr Puzyrov: Conceptualization, Methodology, Software, Writing – original draft. **Jan Awrejcewicz:** Supervision, Investigation, Formal analysis. **Nataliya Losyeva:** Supervision, Software, Validation. **Nina Savchenko:** Visualization, Software, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

This work has been supported by the Polish National Science Center under the grant OPUS 14 No. 2017/27/B/ST8/01330.

References

[1] V. Bolotin, Nonconservative Problems of the Theory of Elastic Stability, Mac-millan, New York, 1963.
[2] H. Ziegler, Die stabilitatskriterien der elastomechanik, Ing. Arch. 20 (1) (1952) 49–56.
[3] M. Abdullatif, R. Mukherjee, A. Hellum, Stabilizing and destabilizing effects of damp-ing in non-conservative systems: Some new results, J. Sound Vib. 413 (2017) 1–14.
[4] J. Awrejcewicz, N. Losyeva, V. Puzyrov, Stability and boundedness of the solutions of multi parameter dynamical systems with circulatory forces, Symmetry 12 (8) (2020) 1210, http://dx.doi.org/10.3390/sym12081210.
[5] A.E. Baikov, P.S. Krasilnikov, The Ziegler effect in a non-conservative mechanical system, J. Appl. Math. Mech. 74 (2010) 51–60.
[6] A. Bloch, P.S. Krishnaprasad, J.E. Marsden, T.S. Ratiu, Dissipation induced instabilities, Ann. Inst. Henri Poincare Anal. Non Lineaire 11 (1994) 37–90.
[7] A. Guran, R.H. Plaut, Stability of Ziegler's pendulum with eccentric load and load-dependent stiffness, Arch. Appl. Mech. 63 (1993) 170–175.
[8] G. Herrmann, I.C. Jong, On the destabilizing effect of damping in nonconservative elastic systems, J. Appl. Mech. 32 (3) (1965) 592–597.
[9] G. Herrmann, Stability of equilibrium of elastic systemss ubjected to non-conservative forces, Appl. Mech. Rev. 20 (1967) 103–108.
[10] J.-D. Jin, Y. Matsuzaki, Bifurcation analysis of double pendulum with a follower force, J. Sound Vib. 154 (2) (1992) 191–204.
[11] J. Kang, C. Krousgrill, F. Sadeghi, Dynamic instability of a thin circular plate with friction interface and its application to disc brake squeal, J. Sound Vib. 316 (1) (2008) 164–179.
[12] O.N. Kirillov, A.P. Seyranian, Stabilization and destabilization of a circulatory system by small velocity-dependent forces, J. Sound Vib. 283 (2005) 781–800.
[13] O.N. Kirillov, F. Verhulst, Paradoxes of dissipation-induced destabilization or who opened Whitney's umbrella? Z. Angew. Math. Mech. 90 (6) (2010) 462–488.
[14] W. Kurnik, P. Przybylowicz, R. Bogacz, Bigoni and Noselli experiment: Is it evidence for flutter in the Ziegler column? Arch. Appl. Mech. 88 (1–2) (2017) 203–213.
[15] J. Lebret, E. Absi, A. Rigolot, About the stability of nonconservative undamped elastic systems: Some new elements, Int. J. Struct. Stab. Dyn. 9 (2) (2009) 357–367.
[16] L.G. Lobas, V. Yu Ichanskii, Limit cycles of a double pendulum subject to a follower force, Internat. Appl. Mech. 45 (6) (2009) 670–678.
[17] L.G. Lobas, Generalized mathematical model of an inverted multilink pendulum with follower force, Internat. Appl. Mech. 41 (5) (2005) 566–572.
[18] R. Plaut, A new destabilization phenomenon in nonconservative systems, ZAMM Z. Angew. Math. Mech. 51 (4) (1971) 319–321.
[19] O.M. O'Reilly, N.K. Malhotra, N.S. Namachchivaya, Some aspects of destabilization in reversible dynamical systems with application to follower forces, Nonlinear Dynam. 10 (1) (1996) 63–87.
[20] J.J. Thomsen, Chaotic dynamics of the partially follower-loaded elastic double pendulum, J. Sound Vib. 188 (3) (1995) 385–405.
[21] A.N. Kounadis, On the paradox of the destabilizing effect of damping in non-conservative systems, Int. J. Non-Linear Mech. 27 (4) (1992) 597–609.
[22] V.V. Koval'chuk, V.L. Lobas, Divergent bifurcations of a double pendulum under the action of an asymmetric follower force, Internat. Appl. Mech. 40 (7) (2004) 821–827.
[23] L.G. Lobas, V.V. Koval'chuk, Influence of the nonlinearity of the elastic elements on the stability of a double pendulum with follower force in the critical case 41 (4) (2005) 455–461.
[24] P. Hagedorn, On the destabilizing effect of non-linear damping in non-conservative systems with follower forces, Int. J. Non-Linear Mech. 5 (2) (1970) 341–358.
[25] C. Semler, H. Alighanbari, M. Paidoussis, A physical explanation of the destabilizing effect of damping, J. Appl. Mech. 65 (3) (1998) 642–648.
[26] J.-J. Sinou, L. Jezequel, Mode coupling instability in friction-induced vibrations and its dependency on system parameters including damping, Eur. J. Mech. A Solids 26 (2007) 106–122.
[27] Y. Sugiyama, M. Langthjem, Physical mechanism of the destabilizing effect of damping in continuous non-conservative dissipative systems, Int. J. Non-Linear Mech. 42 (1) (2007) 132–145.
[28] J. Hulten, Friction phenomena related to drum brake squeal instabilities, in: ASME Design Engineering Technical Conferences, CA, ASME Paper DETC97/VIB-4161, Sacramento, 1997.
[29] R. Krechetnikov, J.E. Marsden, Dissipation induced instabilities in finite dimensions, Rev. Modern Phys. 79 (2) (2007) 519–553.
[30] Y.D. Selyutskiy, On dynamics of an aeroelastic system with two degrees of freedom, Appl. Math. Model. 67 (2019) 449–455.
[31] M.A. Langthjem, Y. Sugiyama, Dynamic stability of columns subjected to follower loads: A survey, J. Sound Vib. 238 (2000) 809–851.
[32] A. Luongo, F. D'Annibale, On the destabilizing effect of damping on discrete and continuous circulatory systems, J. Sound Vib. 333 (2014) 6723–6741.
[33] P. Hagedorn, E. Heffel, P. Lancaster, P.C. Muller, S. Kapuria, Some recent results on MDGKN-systems, ZAMM Z. Angew. Math. Mech. 95 (7) (2015) 695–702.
[34] D. Jekel, P. Hagedorn, Stability of weakly damped MDGKN-systems: The role of velocity proportional terms, ZAMM Z. Angew. Math. Mech. 97 (9) (2017) 1128–1135.
[35] A. Luongo, M. Ferretti, F. D'Annibale, Paradoxes in dynamic stability of mechanical systems: Investigating the causes and detecting the nonlinear behaviors, Springerplus 5 (60) (2016) http://dx.doi.org/10.1186/s40064-016-1684-9.
[36] O.N. Kirillov, D. Bigoni, Classical results and modern approaches to nonconservative stability, in: Dynamic Stability and Bifurcation in Nonconservative Mechanics, CISM Int. Centre for Mech. Sci. Springer, 2019, pp. 129–190.
[37] F. D'Annibale, M. Feretti, On the effects of linear damping on the nonlinear Ziegler's column, Nonlinear Dynam. (2020) http://dx.doi.org/10.1007/s11071-020-05797-y.
[38] A. Lienard, M.H. Chipart, Sur le signe de la partie reelle des racines d'une equation algebrique, J. Math. Pures Appl. 10 (6) (1914) 291–346.
[39] Y.D. Selyutskiy, Potential forces and alternation of stability character in non-conservative systems, Appl. Math. Model. 90 (2021) 191–199.